Multi-Dimensional Stockwell Transforms and Applications

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Contents

1 Introduction

Let f be a signal in $L^2(\mathbb{R})$. Then we can recover frequency information contained in the signal using the Fourier transform $\mathrm{F}f$ given by

$$(\mathrm{F}f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx, \qquad \xi \in \mathbb{R}.$$

Sometimes it is not enough to have information regarding the frequency content of the signal f. In these cases it is useful to look at the Gabor transform $\mathrm{V}_\varphi f$ of the signal with respect to a certain window φ in $L^2(\mathbb{R})$ given by

$$\left(\mathrm{V}_\varphi f\right)(b,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)\overline{\varphi(x-b)}\,dx$$

for all $b, \xi \in \mathbb{R}$. We can notice that

$$\left(\mathrm{V}_\varphi f\right)(b,\xi) = (2\pi)^{-1/2} \left(f, \mathrm{M}_\xi \mathrm{T}_{-b}\varphi\right)_{L^2(\mathbb{R})}$$

for all $b, \xi \in \mathbb{R}$, where $(\,,\,)_{L^2(\mathbb{R})}$ is the inner product in $L^2(\mathbb{R})$, M_ξ and T_{-b} are the modulation operator and the translation operator given by

$$\left(\mathrm{M}_\xi f\right)(x) = e^{ix\xi} f(x), \qquad x \in \mathbb{R},$$

and

$$(\mathrm{T}_{-b} f)(x) = f(x-b), \qquad x \in \mathbb{R},$$

for all measurable functions f on $\mathbb{R}$.

The usefulness of the Gabor transforms in signal analysis is enhanced by the following resolution of the identity formula, which allows the reconstruction of a signal from its Gabor transform.

Theorem 1.1. *Suppose that $\|\varphi\|_{L^2(\mathbb{R})} = 1$, where $\|\,\|_{L^2(\mathbb{R})}$ is the norm in $L^2(\mathbb{R})$. Then for all f and g in $L^2(\mathbb{R})$,*

$$(f,g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\mathrm{V}_\varphi f\right)(b,\xi)\, \overline{\left(\mathrm{V}_\varphi g\right)(b,\xi)}\,db\,d\xi. \tag{1.1}$$

Another way of looking at Theorem 1.1 is that for all f in $L^2(\mathbb{R})$,

$$f = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} (f, \mathrm{M}_\xi \mathrm{T}_{-b} \varphi)_{L^2(\mathbb{R}^n)} \mathrm{M}_\xi \mathrm{T}_{-b} \varphi \, db \, d\xi.$$

The formula in Theorem 1.1 is also known as a continuous inversion formula for the Gabor transform and it is related to the representations of the Heinsenberg group $\mathbb{H}$.

We can use (1.1) to define localization operators $L_{F,\varphi}$ with symbol $F \in L^p(\mathbb{R})$ and window φ associated to the Gabor transform by

$$\left(L_{F,\varphi} f, g\right)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} F(b,\xi) \left(\mathrm{V}_\varphi f\right)(b,\xi) \overline{\left(\mathrm{V}_\varphi g\right)(b,\xi)} \, db \, d\xi.$$

These operators are studied in detail in [32, 33].

In signal analysis, the term $\left(\mathrm{V}_\varphi f\right)(b,\xi)$ gives the time-frequency content of a signal f at time b and frequency ξ by placing the window φ at time b.

It is important to point out that we can extend the Gabor transform here defined to the multi-dimensional case. In fact it is sufficient to look at

$$\begin{aligned}\left(\mathrm{V}_\varphi f\right)(b,\xi) &= (2\pi)^{-n/2} \left(f, \mathrm{M}_\xi \mathrm{T}_{-b} \varphi\right)_{L^2(\mathbb{R}^n)} \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}} e^{-ix\xi} f(x) \overline{\varphi(x-b)} \, dx, \qquad b, \xi \in \mathbb{R}^n.\end{aligned}$$

The drawback here is that a window of fixed width is used for all time b. It is more accurate and desirable if we can have an adaptive window that gives a wide window for low frequency and a narrow window for high frequency. That this can be done comes from familiarity with the wavelet transform that we now recall.

This can be done by looking at the wavelet transform $\mathrm{W}_\varphi f$ of the signal $f \in L^2(\mathbb{R})$ with respect to a certain window φ in $L^2(\mathbb{R})$ given by

$$\left(\mathrm{W}_\varphi f\right)(b,a) = a^{-1/2} \int_{\mathbb{R}} f(x) \overline{\varphi\left(\frac{x-b}{a}\right)} \, dx$$

for all $b \in \mathbb{R}$, $a \in \mathbb{R}_+$. We can notice that

$$\left(\mathrm{W}_\varphi f\right)(b,a) = (f, \mathrm{T}_{-b} \mathrm{D}_{2,a} \varphi)_{L^2(\mathbb{R}^n)}$$

for all $b \in \mathbb{R}$, $a \in \mathbb{R}_+$, where $\mathrm{D}_{2,a}$ is the dilation operator given by

$$(\mathrm{D}_{2,a} f)(x) = a^{-1/2} f\left(\frac{x}{a}\right), \qquad x \in \mathbb{R},$$

for all measurable functions f on $\mathbb{R}$.

The nucleus of the analysis of the wavelet transform is the following resolution of the identity formula, which is a continuous inversion formula.

Theorem 1.2. *Let φ be a nonzero function in $L^2(\mathbb{R})$ such that*

$$c_\varphi = 2\pi \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty. \tag{1.2}$$

Then for all functions f and g in $L^2(\mathbb{R})$,

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\mathrm{W}_\varphi f\right)(b,a) \overline{\left(\mathrm{W}_\varphi g\right)}(b,a) \frac{db\,da}{a^2}.$$

Notice that (1.2) is also known as admissibility condition for wavelets. A nonzero function $\varphi \in L^2(\mathbb{R}^n)$ satisfying (1.2) is called a mother affine wavelet. The adjective *affine* comes from the connection with the affine group $\mathbb{A}$ that is the underpinning of the wavelet transforms. See Chapter 18 of [33] in this connection.

A multi-dimensional version of the wavelet transforms has been introduced in [27] and is given by

$$\begin{aligned}\left(\mathrm{W}_\varphi f\right)(b,a,R) &= (f, \mathrm{T}_{-b}\,\mathrm{D}_{2,aR}\,\varphi)_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{a^{n/2}} \int_{\mathbb{R}^n} f(x) \overline{\varphi\left(\frac{1}{a} R^{-1}(x-b)\right)}\, dx\end{aligned}$$

for all $(b,a,R) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathrm{SO}(n,\mathbb{R})$, where $D_{2,aR}$ is the dilation operator given by

$$(D_{2,aR} f)(x) = a^{-n/2} f\left(a^{-1} R^{-1} x\right), \qquad x \in \mathbb{R}^n,$$

for all measurable functions f on $\mathbb{R}^n$.

Now, let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, combining the merits of the Gabor transform and the wavelet transform, the Stockwell transform $S_\varphi f$ with window φ of a signal f in $L^2(\mathbb{R})$ is defined by

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2} |\xi| \int_{\mathbb{R}} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))}\, dx, \tag{1.3}$$

for all $b \in \mathbb{R}$ and $\xi \in \mathbb{R}\backslash\{0\}$. We note that for all f in $L^2(\mathbb{R})$, all b in $\mathbb{R}$ and all ξ in $\mathbb{R}\backslash\{0\}$,

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2} (f, \mathrm{M}_\xi\, \mathrm{T}_{-b}\, \mathrm{D}_{1,\frac{1}{\xi}}\, \varphi)_{L^2(\mathbb{R})},$$

where the dilation operator $D_{1,\frac{1}{\xi}}$ is defined by

$$(D_{1,\frac{1}{\xi}} f)(x) = |\xi| f(\xi x)$$

for all x in $\mathbb{R}$ and all measurable functions f on $\mathbb{R}$. Besides the modulation with respect to frequency ξ, a notable feature in the Stockwell transform is the normalizing factor in the dilation operator, which is $|\cdot|$ in lieu of $|\cdot|^{1/2}$ as in the case of the wavelet transform, and is the mathematical underpinning of the *absolutely referenced phase information* in [31]. These features distinguish the Stockwell transform from the wavelet transform. Notwithstanding these differences, we have the following formula in [31] relating the Stockwell transform to the Morlet wavelet transform W_ψ.

Theorem 1.3. *For all f in $L^2(\mathbb{R})$,*

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2} e^{-ib\xi} \sqrt{|\xi|} (\Omega_\psi f)(b, 1/\xi)$$

for all $b \in \mathbb{R}$ and $\xi \in \mathbb{R}\backslash\{0\}$, where

$$\psi(x) = e^{ix}\varphi(x)$$

for all x in $\mathbb{R}$.

The Stockwell transform is closely related to the wave packet transform of Cordoba and Fefferman [10], which also involves a combination of translations, modulations and dilations. It should also be mentioned that transforms closely related to the wavelet transforms and the metaplectic representation abound and can be found in the monographs [14], and the works [9, 2].

Of particular importance in the Stockwell transform is the phase correction in the preceding formula given by $e^{-ib\xi}$, which is caused by the phase function $e^{-ix\xi}$ inside the integral defining the Stockwell transform. It is crucial to note that this function picks out the frequency to be localized, but is not translated with respect to time b as is always done for the Morlet wavelet transform [17]. To see the full significance of this, we note that in real-life applications, signals f and windows φ are real-valued functions. Therefore information about the phase $\arg(S_\varphi f)(b,\xi)$ of the Stockwell transform $(S_\varphi f)(b,\xi)$ at time b and frequency ξ comes from the term $e^{-ix\xi}$ at time $b = 0$. But in the case of the Morlet wavelet transform, the phase information is obtained by referencing the windowed signal f with respect to $e^{-i(x-b)\xi}$. This is precisely the *absolutely referenced phase information* in [13] and is responsible for the continuous inversion formula in Theorem 1.4 given later. Another point is that the Stockwell transform is reminiscent of the Morlet wavelet transform, but the applicability of the computational techniques available for the Morlet wavelet transforms is undermined by the inversion $a = 1/\xi$.

The Stockwell transform has recently been used in geophysics [13, 31] and in medical imaging [16, 35]. More recent applications in imaging are in [18, 19]. In view of its versatility, an attempt in understanding the mathematical underpinnings of the Stockwell transform is worthwhile. The following continuous inversion formula for the Stockwell transform can be found in [13] and [34] for the case when

$$\varphi(x) = e^{-\pi x^2}$$

for all x in $\mathbb{R}$.

Theorem 1.4. *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} \varphi(x)\,dx = 1.$$

Then for all f in $L^2(\mathbb{R})$,

$$f = \mathrm{F}^{-1} A S_\varphi f,$$

where F^{-1} is the inverse Fourier transform and A is the time average operator given by

$$(AF)(\xi) = \int_{\mathbb{R}} F(b,\xi)\,db$$

for all ξ in $\mathbb{R}$ and all measurable functions F on $\mathbb{R} \times \mathbb{R}$, provided that the integral exists.

Another continuous inversion formula for the Stockwell transform akin to the continuous inversion formulas for the Gabor transform and the wavelet transform is given by the following theorem.

Theorem 1.5. *Let $\varphi \in L^2(\mathbb{R})$ be such that*

$$c_\varphi = \int_{\mathbb{R}} |\hat{\varphi}(\xi - 1)|^2 \frac{d\xi}{|\xi|} < \infty. \tag{1.4}$$

Then for all f and g in $L^2(\mathbb{R})$,

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(S_\varphi f\right)(b,\xi)\, \overline{\left(S_\varphi g\right)(b,\xi)}\, db\, \frac{d\xi}{|\xi|}.$$

Remark 1.6. Theorem 1.5 tells us that every signal f can be resconstructed from its Stockwell spectrum by means of the formula

$$f = \frac{1}{c_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(S_\varphi f\right)(b,\xi) \left(M_\xi T_{-b} D_{1,\frac{1}{\xi}} \varphi\right) db\, \frac{d\xi}{|\xi|}.$$

Equation (1.4) means that $\hat{\varphi}(-1) = 0$ whenever $\hat{\varphi}$ is continuous at -1. So, it is important to observe that the Gaussian window used exclusively for the Stockwell transform in the literature is *not admissible*.

The aim of this work is to provide a theoretical setting in which to study the Stockwell transforms. We introduce multi-dimensional Stockwell transforms and we give the corresponding continuous inversion formulas analogous to Theorem 1.4 and Theorem 1.5 for the 1-dimensional Stockwell transforms. We elucidate a connection between the affine Heisenberg group $\mathbb{AH}$ and the multi-dimensional Stockwell transforms. Furthermore, we extend the results given in the one-dimensional case regarding the $L^p(\mathbb{R}^n)$-boundedness of the localization operators and the instantaneous frequency.

After Chapter 2 on notation, in Chapter 3 we give a brief introduction on group representation theory and its relations with time-frequency analysis. In particular we are interested in unitary irreducible and square integrable representations of groups or homogenous spaces. In this chapter we introduce and study the affine group $\mathbb{A}$, the reduced Heisenberg group $\mathbb{H}$ and the affine Heisenberg group $\mathbb{AH}$. In Chapter 4 we define the multi-dimensional Stockwell transform and we show its relations with the Gabor transform and the Moritoh wavelet transform. The role played by a certain section of the affine Heisenberg $\mathbb{AH}$ group in defining the Stockwell transform is mentioned too. In Chapter 5 we prove continuous inversion formulas for multi-dimensional Stockwell transforms under different sets of hypotheses. Moreover, we provide a couple of interesting examples of Stockwell transforms that satisfy these set of hypotheses. The results in this chapter extend those in [24] and [29]. Chapter 6 is devoted to the study of the localization operators associated with multi-dimensional Stockwell transforms. We write an explicit relation between these operators and the Weyl transform. This chapter is an extension of the results given in [23]. In Chapter 7 we extend the results given in [19] for the 1-dimensional Stocwkell transform regarding the instantaneous frequency of the signal.

2 Notation

Throughout all this paper we use the following operators on $L^2(\mathbb{R}^n)$.

- Let $\xi \in \mathbb{R}^n$, then the modulation operator M_ξ is given by

$$\left(\mathrm{M}_\xi f\right)(x) = e^{ix\cdot\xi} f(x), \qquad \forall x \in \mathbb{R}^n.$$

- Let $b \in \mathbb{R}^n$, then the translation operator T_{-b} is given by

$$\left(\mathrm{T}_{-b} f\right)(x) = f(x-b), \qquad \forall x \in \mathbb{R}^n.$$

- Let $A \in \mathrm{GL}(n,\mathbb{R})$, then the dilation operator $\mathrm{D}_{s,A}$ is given by

$$\left(\mathrm{D}_{s,A} f\right)(x) = |\det A|^{-1/s} f\left(A^{-1}x\right), \quad \forall x \in \mathbb{R}^n.$$

  In the following we will often write D_A instead of $\mathrm{D}_{2,A}$.

Definition 2.1. We define the **Fourier transform** $\mathrm{F} f$ of the signal f by

$$(\mathrm{F} f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

for all $\xi \in \mathbb{R}^n$.

Notice that $\mathrm{F}, \mathrm{M}_\xi, \mathrm{T}_{-b}$ and D_A are unitary operators on $L^2(\mathbb{R}^n)$. They obey the following composition rules:

$$\begin{array}{lll}
\mathrm{F}\,\mathrm{M}_\xi = \mathrm{T}_{-\xi}\,\mathrm{F}, & \mathrm{F}\,\mathrm{T}_{-b} = \mathrm{M}_{-b}\,\mathrm{F}, & \mathrm{F}\,\mathrm{D}_A = \mathrm{D}_{(A^{-1})^t}\,\mathrm{F}, \\
\mathrm{M}_\xi\,\mathrm{F} = \mathrm{F}\,\mathrm{T}_\xi, & \mathrm{M}_\xi\,\mathrm{T}_{-b} = e^{ib\cdot\xi}\,\mathrm{T}_{-b}\,\mathrm{M}_\xi, & \mathrm{M}_\xi\,\mathrm{D}_A = \mathrm{D}_A\,\mathrm{M}_{A^t\xi}, \\
\mathrm{T}_{-b}\,\mathrm{F} = \mathrm{F}\,\mathrm{M}_b, & \mathrm{T}_{-b}\,\mathrm{M}_\xi = e^{-ib\cdot\xi}\,\mathrm{M}_\xi\,\mathrm{T}_{-b}, & \mathrm{T}_{-b}\,\mathrm{D}_A = \mathrm{D}_A\,\mathrm{T}_{-A^{-1}b}, \\
\mathrm{D}_A\,\mathrm{F} = \mathrm{F}\,\mathrm{D}_{(A^{-1})^t}, & \mathrm{D}_A\,\mathrm{M}_\xi = \mathrm{M}_{(A^{-1})^t\xi}\,\mathrm{D}_A, & \mathrm{D}_A\,\mathrm{T}_{-b} = \mathrm{T}_{-Ab}\,\mathrm{D}_A.
\end{array}$$

Definition 2.2. We define the **Gabor transform** $\mathrm{V}_\varphi f$ of the signal f with respect to the window φ by

$$\left(\mathrm{V}_\varphi f\right)(b,\xi) = (2\pi)^{-n/2} \left(f, \mathrm{M}_\xi\,\mathrm{T}_{-b}\,\varphi\right)_{L^2(\mathbb{R}^n)}$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \overline{\varphi(x-b)} dx$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.3. We define the **wavelet transform** $\mathrm{W}_\varphi f$ of the signal f with respect to the window φ by

$$\begin{aligned}\left(\mathrm{W}_\varphi f\right)(b,a,R) &= \left(f, \mathrm{T}_{-b}\, \mathrm{D}_{2,aR}\, \varphi\right)_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{a^{n/2}} \int_{\mathbb{R}^n} f(x) \overline{\varphi\left(\frac{1}{a} R^{-1}(x-b)\right)} dx\end{aligned}$$

for all $(b,a,R) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathrm{SO}(n,\mathbb{R})$.

Definition 2.4. Let $R : \mathbb{R}^n \ni \xi \mapsto R(\xi) = R_\xi \in \mathrm{SO}(n,\mathbb{R})$, then we define the **Moritoh wavelet transform** $\mathrm{W}_{\frac{1}{|\xi|}R^{-1},\varphi} f$ of the signal f with respect to the window φ by

$$\begin{aligned}\left(\mathrm{W}_{\frac{1}{|\xi|}R^{-1},\varphi} f\right)(b,\xi) &= |\xi|^{n/2} \int_{\mathbb{R}^n} f(x) \overline{\varphi\left(|\xi| R_\xi (x-b)\right)} dx \\ &= \left(f, \mathrm{T}_{-b}\, \mathrm{D}_{2,\frac{1}{|\xi|}R_\xi^{-1}}\, \varphi\right)_{L^2(\mathbb{R}^n)}\end{aligned}$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.5. We define the **Wigner transform** $\mathrm{Wig}(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ by

$$(\mathrm{Wig}(f,g))(b,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f\left(b + \frac{x}{2}\right) \overline{g\left(b - \frac{x}{2}\right)} dx$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.6. We define the **Weyl transform** $\mathrm{W}_\sigma f$ of f in $L^2(\mathbb{R}^n)$ by

$$(\mathrm{W}_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(b,\xi) (\mathrm{Wig}(f,g))(b,\xi)\, db\, d\xi,$$

for all f and g in $L^2(\mathbb{R}^n)$.

Definition 2.7. Let $\{G, \circ_G\}$ and $\{H, \circ_H\}$ be two groups and let $\varphi : H \to Aut(G)$. Then $\left\{G \times H, \circ_\varphi\right\}$, where

$$(g,h) \circ_\varphi (g',h') = \left(g \circ_G \varphi(h) g', h \circ_H h'\right),$$

is a group. This group is called **semi-direct product of** G **and** H and it is denoted by $G \rtimes_\varphi H$.

Definition 2.8. Let X be an infinite-dimensional complex and separable Hilbert space. Let

$A : X \to X$ be a compact operator. Then the operator $|A| : X \to X$ defined by

$$|A| = \sqrt{A^* A}$$

is positive and compact. So there exists for X an orthonormal basis $\{\varphi_k\}$ consisting of eigenvectors of $|A|$. Let s_k be the eigenvalue of $|A| : X \to X$ corresponding to the eigenvector φ_k. Then we say that the compact operator $A : X \to X$ is in the **Schatten-von Neumann class** S_p, $1 \leqslant p < \infty$ if

$$\sum_{k=1}^{\infty} s_k^p < \infty.$$

By convention, S_∞ is taken to be simply the C^*-algebra of all bounded and linear operator on X. The Schatten-von Neumann class S_1 is also known as the **trace class**. Given an operator $A : X \mapsto X$ in the trace class we can define its trace $\operatorname{tr}(A)$ as

$$\sum_{k=1}^{\infty} (A\varphi_k, \varphi_k)_X ,$$

where $\{\varphi_k\}_k$ is an orthonormal basis for X.

3 Group representation theory

3.1 Localization operators and homogeneous spaces

Let X be a locally compact and Hausdorff topological space and let G be a locally compact and Hausdorff group. We say that G is a left transformation group on X if there exists a continuous mapping $G \times X \ni (g, x) \mapsto gx \in X$, such that for all $g \in G$, the mapping $X \ni x \mapsto gx \in X$ is a homeomorphism of X onto X,

$$(gh)x = g(hx), \qquad g, h \in G,\ x \in X,$$

and

$$ex = x, \qquad x \in X,$$

where e is the identity element in G. The topological space X on which G acts is called a G-space and G is sometimes called a group action on X. Let G be a left transformation group on X such that for all x_1 and x_2 in X there exists an element $g \in G$ for which $x_2 = gx_1$. Then we say that the action of G on X is transitive and we say that X is a homogeneous space.

Proposition 3.1. *Let X be a homogeneous space on which G acts transitively and let $x \in X$. Then the set H_x defined by*

$$H_x = \{g \in G : gx = x\},$$

is a closed subgroup of G. We call H_x the stability subgroup of G associated to x.

Let G be a locally compact and Hausdorff group and let H be a closed subgroup of G. Let

$$X = G/H = \{gH : g \in G\},$$

and gH, $g \in G$, is the left coset of g in H. Then the action of G on X defined by

$$G \times X \ni (g, hH) \longmapsto (gh)H \in X, \qquad g, h \in G,$$

is transitive. Hence G/H is a homogeneous space.

Let G/H be a homogeneous space, where G is a locally compact and Hausdorff group and H is a closed subgroup of G and let μ be a Borel measure on G/H. A Borel measure μ on X is said to be left quasi-invariant if μ and μ_g are equivalent measures on X, where

$$\mu_g(S) = \mu(gS), \qquad g \in G,$$

for all Borel subsets S of G/H. Notice that we can always equip G/H with a left quasi invariant measure.

We call the canonical surjection of G on G/H the mapping $q : G \to G/H$ defined by

$$G \ni g \mapsto gH \in G/H, \qquad g \in G.$$

A mapping $\sigma : G/H \to G$ such that by

$$q(\sigma(x)) = x, \qquad x \in G/H,$$

is said to be a section on G/H. Assume that G admits a unitary representation π of G on $L^2(\mathbb{R}^n)$, *i.e.*,

$$\pi : G \to U\left(L^2(\mathbb{R}^n)\right),$$

where $U(L^2(\mathbb{R}^n))$ is the group of all unitary operators on $L^2(\mathbb{R}^n)$. Suppose that there exists an element $\varphi \in L^2(\mathbb{R}^n)$ such that

$$\int_{G/H} \left|(\pi(\sigma(x))\varphi, f)\right|^2 d\mu(x) < \infty, \qquad \forall f \in L^2(\mathbb{R}^n).$$

Then we say that σ is a strictly admissible section and φ is an admissible wavelet.

Remark 3.2. Notice that this definition can be found in [21]. Some authors, for example [33], talks about square integrable sections.

We define the constant $c_{\sigma,H,\varphi}$, by

$$c_{\sigma,H,\varphi} = \int_{G/H} \left|(\pi(\sigma(x))\varphi, \varphi)\right|^2 d\mu(x).$$

Let $F \in L^1(G/H)$,then we define the linear operator $L_{F,\sigma,H,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\left(L_{F,\sigma,H,\varphi} f_1, f_2\right) = \frac{1}{c_{\sigma,H,\varphi}} \int_{G/H} F(x)\,(f_1, \pi(\sigma(x))\varphi)\,(\pi(\sigma(x))\varphi, f_2)\, d\mu(x), \qquad (3.1)$$

for all f_1 and f_2 in $L^2(\mathbb{R}^n)$.

Proposition 3.3. *Let σ be a strictly admissible section. Then the localization operator*

$L_{F,\sigma,H,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and

$$\left|\left|L_{F,\sigma,H,\varphi}\right|\right|_{B(L^2(\mathbb{R}^n))} \leqslant \frac{1}{c_{\sigma,H,\varphi}} \|F\|_{L^1(G/H)}.$$

Proposition 3.4. *Let σ be a strictly admissible section. Then the localization operator* $L_{F,\sigma,H,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in the trace class S_1, and its trace is given by

$$\operatorname{tr}\left(L_{F,\sigma,H,\varphi}\right) = \frac{1}{c_{\sigma,H,\varphi}} \int_{G/H} F(x) d\mu(x).$$

Remark 3.5. Notice that Proposition 3.3 and Proposition 3.4 are in fact Proposition 25.1, Proposition 25.3, and Proposition 25.4 on pp.143-145 in [33].

3.2 Affine group

We denote with $\mathbb{A}$ the affine group, $i.e.$

$$\mathbb{A} = \mathbb{R} \rtimes \mathbb{R}_+ = \{\mathbb{R} \times \mathbb{R}_+, \circ_{\mathbb{A}}\}.$$

Given (b,a) and (b',a') in $\mathbb{A}$, then

$$(b,a) \circ_{\mathbb{A}} (b',a') = (b + ab', aa'),$$

and

$$(b,a)^{-1} = \left(-\frac{b}{a}, \frac{1}{a}\right).$$

Proposition 3.6. *The left and right Haar measures on $\mathbb{A}$ are given by*

$$d\mu = \frac{db\,da}{a^2}$$

and

$$d\nu = \frac{db\,da}{a}$$

respectively.

Proof. To prove left invariance, let f be an integrable function on $\mathbb{A}$ with respect to the $d\mu$.

Then, for all (b', a') in $\mathbb{A}$, we get

$$\int_{\mathbb{A}} f\left(\left(b', a'\right) \circ_{\mathbb{A}} (b, a)\right) d\mu = \int_{\mathbb{R}_+} \int_{\mathbb{R}} f\left(b' + a' b, a' a\right) \frac{db\, da}{a^2}.$$

Setting $\beta = b' + a' b$ and $\alpha = a' a$, we have

$$\begin{aligned}
&\int_{\mathbb{A}} f\left(\left(b', a'\right) \circ_{\mathbb{A}} (b, a)\right) d\mu \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} f\left(b' + a' b, a' a\right) \frac{db\, da}{a^2} \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(\beta, \alpha) \frac{d\beta\, d\alpha}{\alpha^2} \\
&= \int_{\mathbb{A}} f(b, a)\, d\mu.
\end{aligned}$$

Analogously we can prove right invariance for $d\nu$.

Remark 3.7. The affine group $\mathbb{A}$ is a locally compact non unimodular Hausdorff group.

We can introduce the representation

$$\pi_{\mathbb{A}}(b, a)\,\varphi = \mathrm{T}_{-b}\, \mathrm{D}_{2,a}\, \varphi.$$

Remark 3.8. Notice that we can write

$$\left(\mathrm{W}_{\varphi} f\right)(b, a) = \left(f, \pi_{\mathbb{A}}(b, a)\,\varphi\right)_{L^2(\mathbb{R}^n)}$$

for all $(b, a) \in \mathbb{A}$.

Theorem 3.9. *$\pi_{\mathbb{A}}$ is a unitary irreducible and square-integrable representation of $\mathbb{A}$ on the Hardy space $H^2_+(\mathbb{R})$.*

3.3 Similitude group

We denote with $\mathrm{SIM}(n)$ the n-dimensonal similitude group, *i.e.*,

$$\mathrm{SIM}(n) = \mathbb{R}^n \rtimes (\mathbb{R}_+ \times \mathrm{SO}(n, \mathbb{R})) = \{\mathbb{R}^n \times \mathbb{R}_+ \times \mathrm{SO}(n, \mathbb{R}), \circ_{\mathrm{SIM}(n)}\}.$$

Given (b, a, R) and (b', a', R') in $\mathrm{SIM}(n)$, then

$$(b, a, R) \circ_{\mathrm{SIM}(n)} \left(b', a', R'\right) = \left(b + aRb', aa', RR'\right),$$

and

$$(b,a,R)^{-1} = \left(-(aR)^{-1}\, b, a^{-1}, R^{-1}\right).$$

Proposition 3.10. *Let* $dm(R)$ *be the Haar measure on* $\mathrm{SO}(n,\mathbb{R})$*, normalized so that* $m(\mathrm{SO}(n,\mathbb{R})) = 1$*. Then the left and right Haar measures on* $\mathrm{SIM}(n)$ *are given by*

$$d\mu = db\, \frac{da}{a^{n+1}}\, dm(R),$$

and

$$d\nu = db\, \frac{da}{a^{n}}\, dm(R),$$

respectively.

Proof. To prove left invariance, let f be an integrable function on $\mathrm{SIM}(n)$ with respect to the $d\mu$. Then, for all (b', a', R') in $\mathrm{SIM}(n)$, we get

$$\begin{aligned}
&\int_{\mathrm{SIM}(n)} f\left(\left(b', a', R'\right) \circ_{\mathrm{SIM}(n)} (b,a,R)\right) d\mu \\
&= \int_{\mathrm{SO}(n,\mathbb{R})} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} f\left(b' + a'R'b, a'a, RR'\right) db\, \frac{da}{a^{n+1}}\, dm(R).
\end{aligned}$$

Setting $\beta = b' + a'R'b$, $\alpha = a'a$ and $Q = R'R$, we have

$$\begin{aligned}
&\int_{\mathrm{SIM}(n)} f\left(\left(b', a', R'\right) \circ_{\mathrm{SIM}(n)} (b,a,R)\right) d\mu \\
&= \int_{\mathrm{SO}(n,\mathbb{R})} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} f\left(b' + a'R'b, a'a, RR'\right) db\, \frac{da}{a^{n+1}}\, dm(R) \\
&= \int_{\mathrm{SO}(n,\mathbb{R})} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} f(\beta, \alpha, Q)\, d\beta\, \frac{d\alpha}{\alpha^{n+1}}\, dm(Q) \\
&= \int_{\mathrm{SIM}(n)} f(b,a,R)\, d\mu.
\end{aligned}$$

Analogously we can prove right invariance for $d\nu$.

Remark 3.11. The n-dimensional similitude group $\mathrm{SIM}(n)$ is a locally compact and non-unimodular Hausdorff group.

We can introduce the representation

$$\pi_{\mathrm{SIM}(n)}(b,a,R)\,\varphi = \mathrm{T}_{-b}\, \mathrm{D}_{2,aR}\, \varphi.$$

Remark 3.12. Notice that we can write

$$\left(\mathrm{W}_{\varphi} f\right)(b,a,R) = \left(f, \pi_{\mathrm{SIM}(n)}(b,a,R)\varphi\right)_{L^2(\mathbb{R}^n)}$$

for all $(b,a,R) \in \mathrm{SIM}(n)$.

Theorem 3.13. *$\pi_{\mathrm{SIM}(n)}$ is a unitary, irreducible and square-integrable representation of $\mathrm{SIM}(n)$ on $L^2(\mathbb{R}^n)$.*

3.4 Weyl-Heisenberg group

We denote by $\mathbb{WH}$ the n-dimensional Weyl-Heisenberg group, $i.e.$,

$$\mathbb{WH} = \{\mathbb{R}^n \times \mathbb{R}^n \times S^1, \circ_{\mathbb{WH}}\}.$$

Given (b,ξ,ϑ) and (b',ξ',ϑ') in $\mathbb{WH}$, then

$$(b,\xi,\vartheta) \circ_{\mathbb{WH}} \left(b',\xi',\vartheta'\right) = \left(b+b', \xi+\xi', \vartheta+\vartheta'+\xi\cdot b'\right),$$

where the third entry in the point is intended $\mod [0,2\pi]$, and

$$(b,\xi,\vartheta,a,R)^{-1} = (-b,-\xi,-\vartheta+b\cdot\xi).$$

Proposition 3.14. *The left and right Haar measures on $\mathbb{WH}$ are equal and is given by*

$$d\mu = d\,b\,d\xi\,d\vartheta.$$

Proof. To prove left invariance, let f be an integrable function on $\mathbb{WH}$ with respect to $d\mu$. For sake of simplicity, let us assume that $f(b,\xi,\cdot)$ is a periodic function with period 2π for fixed but arbitrary b and ξ in $\mathbb{R}^n$. Then, for all (b',ξ',ϑ') in $\mathbb{WH}$, we get

$$\int_{\mathbb{WH}} f\left(\left(b',\xi',\vartheta'\right)\circ_{\mathbb{WH}}(b,\xi,\vartheta)\right)d\mu$$
$$= \int_{S^1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(b+b',\xi+\xi',\vartheta+\vartheta'+\xi\cdot b' \mod [0,2\pi]\right)d\,b\,d\xi\,d\vartheta.$$

Setting $q = b'+b$, $p = \xi'+\xi$ and $\varphi = \vartheta+\vartheta'+\xi\cdot b'$, we have

$$\int_{\mathbb{WH}} f\left(\left(b',\xi',\vartheta'\right)\circ_{\mathbb{WH}}(b,\xi,\vartheta)\right)d\mu$$
$$= \int_{S^1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(b+b',\xi+\xi',\vartheta+\vartheta'+\xi\cdot b' \mod [0,2\pi]\right)d\,b\,d\xi\,d\vartheta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(b+b', \xi+\xi', \vartheta+\vartheta'+\xi\cdot b'\right) db\, d\xi\, d\vartheta \\
&= \int_{\vartheta'+\xi\cdot b'}^{2\pi+\vartheta'+\xi\cdot b'} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(q,p,\varphi)\, dq\, dp\, d\varphi \\
&= \int_{\mathbb{WH}} f(b,\xi,\vartheta)\, d\mu.
\end{aligned}
$$

With similar computations we can check the right invariance of $d\mu$.

Remark 3.15. The n-dimensional Weyl-Heisenberg group $\mathbb{WH}$ is a locally compact, unimodular and Hausdorff group.

We can introduce the representation

$$\pi_{\mathbb{WH}}(b,\xi,\vartheta)\,\varphi = e^{i\vartheta}\,\mathrm{T}_{-b}\,\mathrm{M}_{\xi}\,\varphi.$$

Remark 3.16. Notice that we can write

$$\left(\mathrm{V}_{\varphi} f\right)(b,\xi) = (2\pi)^{-n/2}\, e^{-ib\cdot\xi}\,\left(f, \pi_{\mathbb{WH}}(b,\xi,0)\,\varphi\right)_{L^2(\mathbb{R}^n)}$$

for all $(b,\xi,0) \in \mathbb{WH}$.

Theorem 3.17. $\pi_{\mathbb{WH}}$ is a unitary, irreducible and square-integrable representation of $\mathbb{WH}$ on $L^2(\mathbb{R}^n)$.

3.5 Reduced Heisenberg group

We denote by $\mathbb{H}$ the n-dimensional reduced Heisenberg group, $i.e.$

$$\mathbb{H} = \{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \circ_{\mathbb{H}}\}.$$

Given (b,ξ,ϑ) and (b',ξ',ϑ') in $\mathbb{H}$, then

$$(b,\xi,\vartheta) \circ_{\mathbb{H}} \left(b',\xi',\vartheta'\right) = \left(b+b', \xi+\xi', \vartheta+\vartheta'+\xi\cdot b'\right),$$

and

$$(b,\xi,\vartheta)^{-1} = (-b, -\xi, -\vartheta + b\cdot\xi).$$

Proposition 3.18. *The left and right Haar measures on* $\mathbb{H}$ *are equal and is given by*

$$d\mu = db\, d\xi\, d\vartheta.$$

Proof. To prove left invariance, let f be an integrable function on $\mathbb{H}$ with respect to the $d\mu$. Then, for all (b',ξ',ϑ') in $\mathbb{H}$, we get

$$\begin{aligned}&\int_{\mathbb{H}} f\left(\left(b',\xi',\vartheta'\right)\circ_{\mathbb{H}}(b,\xi,\vartheta)\right)d\mu\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(b+b',\xi+\xi',\vartheta+\vartheta'+\xi\cdot b'\right)db\,d\xi\,d\vartheta.\end{aligned}$$

Setting $q=b'+b$, $p=\xi'+\xi$ and $\varphi=\vartheta+\vartheta'+\xi\cdot b'$, we have

$$\begin{aligned}&\int_{\mathbb{H}} f\left(\left(b',\xi',\vartheta'\right)\circ_{\mathbb{H}}(b,\xi,\vartheta)\right)d\mu\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(b+b',\xi+\xi',\vartheta+\vartheta'+\xi\cdot b'\right)db\,d\xi\,d\vartheta\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(q,p,\varphi)\,dq\,dp\,d\varphi\\ =&\int_{\mathbb{H}} f(b,\xi,\vartheta)\,d\mu.\end{aligned}$$

With similar computations we can check the right invariance of $d\mu$.

Remark 3.19. The n-dimensional Weyl-Heisenberg group $\mathbb{H}$ is a locally compact, unimodular and Hausdorff group.

We can introduce the representation

$$\pi_{\mathbb{H}}(b,\xi,\vartheta)\,\varphi=e^{i\vartheta}\,\mathrm{T}_{-b}\,\mathrm{M}_{\xi}\,\varphi.$$

Remark 3.20. Notice that we can write

$$\left(\mathrm{V}_{\varphi} f\right)(b,\xi)=(2\pi)^{-n/2}\,e^{-ib\cdot\xi}\left(f,\pi_{\mathbb{H}}(b,\xi,0)\,\varphi\right)_{L^2(\mathbb{R}^n)}$$

for all $(b,\xi,0)\in\mathbb{H}$.

Theorem 3.21. *$\pi_{\mathbb{H}}$ is a unitary, irreducible and non-square-integrable representation of $\mathbb{H}$ on $L^2(\mathbb{R}^n)$.*

Theorem 3.22. *Let H be the closed subgroup of $\mathbb{H}$ defined as*

$$H=\{(0,0,\vartheta)\in\mathbb{H}\}.$$

Let $\sigma:\mathbb{H}/H\to\mathbb{H}$ be the Borel section given by

$$\sigma(b,\xi)=(b,\xi,0).$$

Then σ is a strictly admissible section. Furthermore, every non zero $\varphi \in L^2(\mathbb{R}^n)$ is an admissible wavelet for this strictly admissible section, i.e. given $\varphi \in L^2(\mathbb{R}^n)$ such that $\|\varphi\| = 1$, we have

$$c_{\varphi,H,\sigma} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\left(\varphi, \pi_{\mathbb{H}}(\sigma(b,\xi))\varphi\right)\right|^2 db\, d\xi = (2\pi)^n .$$

Remark 3.23. Notice that we can write

$$\left(\mathrm{V}_{\varphi} f\right)(b,\xi) = (2\pi)^{-n/2} e^{-ib\cdot\xi}\left(f, \pi_{\mathbb{H}}(\sigma(b,\xi))\varphi\right)_{L^2(\mathbb{R}^n)}$$

for all $(b,\xi) \in \mathbb{H}/H$.

3.6 Affine Heisenberg group

We denote by $\mathbb{AH}$ the n-dimensonal affine Heisenberg group, $i.e.$

$$\mathbb{AH} = \mathbb{H} \rtimes (\mathbb{R}_+ \times \mathrm{SO}(n,\mathbb{R})) = \{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \times \mathrm{SO}(n,\mathbb{R}), \circ_{\mathbb{AH}}\}.$$

Given (b,ξ,ϑ,a,R) and $(b',\xi',\vartheta',a',R')$ in $\mathbb{AH}$, then

$$\begin{aligned}(b,\xi,\vartheta,a,R) &\circ_{\mathbb{AH}} \left(b',\xi',\vartheta',a',R'\right)\\ &= \left(b + aRb', \xi + a^{-1}R\xi', \vartheta + \vartheta' + \xi\cdot\left(aRb'\right), aa', RR'\right),\end{aligned}$$

and

$$(b,\xi,\vartheta,a,R)^{-1} = \left(-a^{-1}R^{-1}b, -aR^{-1}\xi, -\vartheta + b\cdot\xi, a^{-1}, R^{-1}\right).$$

Proposition 3.24. *Let $dm(R)$ be the Haar measure on $\mathrm{SO}(n,\mathbb{R})$, normalized so that $m(\mathrm{SO}(n,\mathbb{R})) = 1$. Then the left and right Haar measures on $\mathbb{AH}$ are equal and is given by*

$$d\mu = db\, d\xi\, d\vartheta \frac{da}{a} dm(R).$$

Proof. To prove left invariance, let f be an integrable function on $\mathbb{AH}$ with respect to the $d\mu$. Then, for all $(b',\xi',\vartheta',a',R')$ in $\mathbb{AH}$, we get

$$\begin{aligned}&\int_{\mathbb{AH}} f\left(\left(b',\xi',\vartheta',a',R'\right)\circ_{\mathbb{AH}}(b,\xi,\vartheta,a,R)\right)d\mu\\ &= \int_{\mathrm{SO}(n,\mathbb{R})}\int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(b' + a'R'b, \xi' + a'^{-1}R'\xi, \vartheta' + \vartheta + \xi'\cdot\left(a'R'b\right), a'a, R'R\right)\end{aligned}$$

$$\times\, db\, d\xi\, d\vartheta\, \frac{da}{a}\, dm(R).$$

Setting $q = b' + a'R'b$, $p = \xi' + a'^{-1}R'\xi$, $\varphi = \vartheta' + \vartheta + \xi' \cdot (a'R'b)$, $\alpha = a'a$ and $Q = R'R$, we have

$$\begin{aligned}
&\int_{\mathbb{AH}} f\left(\left(b',\xi',\vartheta',a',R'\right) \circ_{\mathbb{AH}} (b,\xi,\vartheta,a,R)\right) d\mu \\
&= \int_{\mathrm{SO}(n,\mathbb{R})} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(b' + a'R'b, \xi' + a'^{-1}R'\xi, \vartheta' + \vartheta + \xi' \cdot \left(a'R'b\right), a'a, R'R\right) \\
&\quad \times db\, d\xi\, d\vartheta\, \frac{da}{a}\, dm(R) \\
&= \int_{\mathrm{SO}(n,\mathbb{R})} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(q,p,\varphi,\alpha,Q)\, dq\, dp\, d\varphi\, \frac{d\alpha}{\alpha}\, dm(Q) \\
&= \int_{\mathbb{AH}} f(b,\xi,\vartheta,a,R)\, d\mu.
\end{aligned}$$

Analogously we can prove the right invariance of $d\mu$.

Remark 3.25. The n-dimensional affine Heisenberg $\mathbb{AH}$ group is a locally compact, unimodular and Hausdorff group.

We can introduce the representation

$$\pi_{\mathbb{AH}}(b,\xi,\vartheta,a,R)\,\varphi = e^{i\vartheta}\, \mathrm{T}_{-b}\, \mathrm{M}_{\xi}\, \mathrm{D}_{2,aR}\, \varphi. \tag{3.2}$$

Theorem 3.26. *$\pi_{\mathbb{AH}}$ is a unitary, irreducible and non-square-integrable representation of $\mathbb{AH}$ on $L^2(\mathbb{R}^n)$.*

Theorem 3.27. *Let $H = \{(0,0,\vartheta,a,R) \in \mathbb{AH}\}$ be a closed subgroup of $\mathbb{AH}$, let $a : \mathbb{R}^n \ni \xi \mapsto a(\xi) \in \mathbb{R}_+$ and $R : \mathbb{R}^n \ni \xi \to R(\xi) \in \mathrm{SO}(n,\mathbb{R})$ be two piecewise differentiable functions such that we can find a fixed $(1,2)$-tensor F and a fixed $(1,1)$-tensor G such that*

$$\left(a(\xi)^{-1} R(\xi)\right)^i_{\;j} = F^i_{jl}\xi^l + G^i_j,$$

and let $\eta_\zeta : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by

$$\eta_\zeta(\xi) = a(\xi)\, R(\xi)^{-1}(\zeta - \xi),$$

is such that $\eta_\zeta(\mathbb{R}^n) = \mathbb{R}^n$, for all $\zeta \in \mathbb{R}^n$. Let $\sigma : \mathbb{AH}/H \to \mathbb{AH}$ be the section given by

$$\sigma(b,\xi) \mapsto (b,\xi,0,a(\xi),R(\xi))$$

and let $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta)\right|^2 \frac{d\eta}{\left|\det\left(F^i_{jk}\eta^j + \delta^i_k\right)\right|} < \infty.$$

Then we have a resolution of the identity formula, $i.e.$*,*

$$c_\varphi (f,g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f, \pi_{\mathbb{AH}}(\sigma(b,\xi)))_{L^2(\mathbb{R}^n)} \overline{(\pi_{\mathbb{AH}}(\sigma(b,\xi)), g)_{L^2(\mathbb{R}^n)}}\, db\, d\xi.$$

Proof. The proof is given in [21].

4 Stockwell transforms

Definition 4.1. Let $1 \leqslant s < \infty$ and $A : \mathbb{R}^n \ni \xi \mapsto A(\xi) = A_\xi \in \mathrm{GL}(n,\mathbb{R})$. Then we define the **Stockwell transform** $\mathrm{S}_{s,A,\varphi} f$ of the signal f with respect to the window φ by

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\, dx$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Remark 4.2. Notice that we can write

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left|\det A_\xi\right|^{1/2-1/s} \left(\mathrm{S}_{2,A,\varphi} f\right)(b,\xi).$$

Remark 4.3. Choosing $n = 1$, $s = 1$ and $A : \mathbb{R} \ni \xi \mapsto 1/\xi \in \mathbb{R}\setminus\{0\}$, we recover the 1-dimensional Stockwell transform defined in 1.3.

We can elucidate the link between the analyzed signal f and the analyzing window φ.

Proposition 4.4. *Let f and φ be signals in $L^2(\mathbb{R}^n)$. Then we have*

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = e^{-ib\cdot\xi}\, \overline{\left(\mathrm{S}_{s,A,f}\, \varphi\right)\left(-A_\xi^{-1} b, -A_\xi^t \xi\right)}, \qquad b,\xi \in \mathbb{R}^n.$$

Proof. It is sufficient to prove the proposition for $s = 2$. In this case we can write

$$\begin{aligned}
\left(\mathrm{S}_{2,A,\varphi} f\right)(b,\xi) &= (2\pi)^{-n/2} \left(f, M_\xi T_{-b} D_{2,A_\xi} \varphi\right)_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} \left(D_{2,A_\xi^{-1}} T_b M_{-\xi} f, \varphi\right)_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} \left(T_{A_\xi^{-1} b} M_{-A_\xi^t \xi} D_{2,A_\xi^{-1}} f, \varphi\right)_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(M_{-A_\xi^t \xi} T_{A_\xi^{-1} b} D_{2,A_\xi^{-1}} f, \varphi\right)_{L^2(\mathbb{R}^n)} \\
&= e^{-ib\cdot\xi}\, \overline{(2\pi)^{-n/2} \left(\varphi, M_{-A_\xi^t \xi} T_{A_\xi^{-1} b} D_{2,A_\xi^{-1}} f\right)_{L^2(\mathbb{R}^n)}} \\
&= e^{-ib\cdot\xi}\, \overline{\left(\mathrm{S}_{2,A,f}\, \varphi\right)\left(-A_\xi^{-1} b, -A_\xi^t \xi\right)}.
\end{aligned}$$

We give here an alternative formulation of Stockwell transforms.

Proposition 4.5. *Let f be a signal in $L^2(\mathbb{R}^n)$ and let φ be a window in $L^2(\mathbb{R}^n)$. Then*

$$\left(\mathrm{F}_{b\mapsto\zeta}\,\mathrm{S}_{s,A,\varphi}\,f\right)(\zeta,\xi)=\left(\mathrm{T}_\xi\,\mathrm{F}f\right)(\zeta)\,\overline{\left(\mathrm{D}_{\frac{s}{s-1},\left(A_\xi^{-1}\right)^t}\,\mathrm{F}\varphi\right)(\zeta)},\qquad \zeta,\xi\in\mathbb{R}^n,$$

which implies

$$\left(\mathrm{S}_{s,A,\varphi}\,f\right)(b,\xi)=\left(\mathrm{M}_{-\xi}\left(f*\mathrm{M}_\xi\,\mathrm{D}_{s,-A_\xi}\,\overline{\varphi}\right)\right)(b),\qquad b,\xi\in\mathbb{R}^n.$$

Proof. Thanks to the Fubini-Tonelli theorem and to the change of variable $y=A_\xi^{-1}(x-b)$, we get

$$\begin{aligned}
&\left(\mathrm{F}_{b\mapsto\zeta}\,\mathrm{S}_{s,A,\varphi}\,f\right)(\zeta,\xi)\\
&=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ib\cdot\zeta}\left(\mathrm{S}_{s,A,\varphi}\,f\right)(b,\xi)\,db\\
&=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ib\cdot\zeta}\,(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\\
&\quad\times\int_{\mathbb{R}^n}f(x)\,e^{-ix\cdot\xi}\,\overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\,dx\,db\\
&=(2\pi)^{-n}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\int_{\mathbb{R}^n}e^{-ib\cdot\zeta}\,\overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\,db\,dx\\
&=(2\pi)^{-n}\left|\det A_\xi\right|^{1-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\int_{\mathbb{R}^n}e^{-i\left(x-A_\xi y\right)\cdot\zeta}\,\overline{\varphi(y)}\,dy\,dx\\
&=(2\pi)^{-n}\left|\det A_\xi\right|^{1-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\,e^{-ix\cdot\zeta}\int_{\mathbb{R}^n}e^{iy\cdot A_\xi^t\zeta}\,\overline{\varphi(y)}\,dy\,dx\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{1-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\,e^{-ix\cdot\zeta}\,\overline{\widehat{\varphi}\left(A_\xi^t\zeta\right)}\,dx\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{1-1/s}\,\overline{\widehat{\varphi}\left(A_\xi^t\zeta\right)}\int_{\mathbb{R}^n}e^{-ix\cdot(\zeta+\xi)}f(x)\,dx\\
&=\left|\det A_\xi\right|^{1-1/s}\,\widehat{f}(\zeta+\xi)\,\overline{\widehat{\varphi}\left(A_\xi^t\zeta\right)}\\
&=\left(\mathrm{T}_\xi\,\mathrm{F}f\right)(\zeta)\,\overline{\left(\mathrm{D}_{\frac{s}{s-1},\left(A_\xi^{-1}\right)^t}\,\mathrm{F}\right)\varphi(\zeta)}\\
&=\left(\mathrm{F}\mathrm{M}_{-\xi}\,f\right)(\zeta)\,\overline{\left(\mathrm{F}\mathrm{D}_{s,A_\xi}\,\varphi\right)(\zeta)}\\
&=\left(\mathrm{F}\mathrm{M}_{-\xi}\,f\right)(\zeta)\left(\mathrm{F}\mathrm{D}_{s,-A_\xi}\,\overline{\varphi}\right)(\zeta),\qquad \zeta,\xi\in\mathbb{R}^n.
\end{aligned}$$

The second formula follows because

$$\left(\mathrm{F}_{b\mapsto\zeta}\,\mathrm{S}_{s,A,\varphi}\,f\right)(\zeta,\xi)=\left(\mathrm{F}\mathrm{M}_{-\xi}\,f\right)(\zeta)\left(\mathrm{F}\mathrm{D}_{s,-A_\xi}\,\overline{\varphi}\right)(\zeta)$$

$$= \left(\mathrm{F} \left(\mathrm{M}_{-\xi} f * \mathrm{D}_{s,-A_\xi} \overline{\varphi} \right) \right) (\zeta), \qquad \zeta, \xi \in \mathbb{R}^n.$$

And via the Fourier inversion formula, we get

$$\begin{aligned} \left(\mathrm{S}_{s,A,\varphi} f \right) (b, \xi) &= \left(\mathrm{M}_{-\xi} f * \mathrm{D}_{s,-A_\xi} \overline{\varphi} \right) (b) \\ &= \left(\mathrm{M}_{-\xi} \left(f * \mathrm{M}_\xi \mathrm{D}_{s,-A_\xi} \overline{\varphi} \right) \right) (b), \qquad b, \xi \in \mathbb{R}^n. \end{aligned}$$

Notice that the Stockwell transform describes the behaviour of the signal f under convolution with a dilated and modulated window φ. In fact,

$$\left| \left(\mathrm{S}_{s,A,\varphi} f \right) (b, \xi) \right| = \left| \left(f * \mathrm{M}_\xi \mathrm{D}_{s,-A_\xi} \overline{\varphi} \right) (b) \right|, \qquad b, \xi \in \mathbb{R}^n.$$

With the new formulation of the Stockwell transforms in place, we can give the following estimate.

Proposition 4.6. *Let φ be in $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Then*

$$\left\| \varphi_{s,b,\xi} \right\|_{L^p(\mathbb{R}^n)} = \left| \det A_\xi \right|^{1/p - 1/s} \left\| \varphi \right\|_{L^p(\mathbb{R}^n)},$$

where

$$\varphi_{s,b,\xi} = \mathrm{M}_\xi \mathrm{T}_{-b} \mathrm{D}_{s,A} \varphi,$$

and

$$\left| \left(\mathrm{S}_{s,A,\varphi} f \right) (b, \xi) \right| \leqslant (2\pi)^{-n/2} \left| \det A_\xi \right|^{1/p - 1/s} \left\| f \right\|_{L^{p'}(\mathbb{R}^n)} \left\| \varphi \right\|_{L^p(\mathbb{R}^n)}, \quad b, \xi \in \mathbb{R}^n,$$

where p' is the conjugated index of p.

Proof. Let $1 \leqslant p < \infty$. Then, thanks to the Fubini-Tonelli theorem and to the change of variable $y = A_\xi^{-1}(x - b)$, we get

$$\begin{aligned} \left\| \varphi_{s,b,\xi} \right\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} \left| \varphi_{s,b,\xi}(x) \right|^p dx \\ &= \int_{\mathbb{R}^n} \left| \mathrm{M}_\xi \mathrm{T}_{-b} \mathrm{D}_{s,A} \varphi(x) \right|^p dx \\ &= \int_{\mathbb{R}^n} \left| e^{ix \cdot \xi} \left| \det A_\xi \right|^{-1/s} \varphi \left(A_\xi^{-1} (x - b) \right) \right|^p dx \\ &= \int_{\mathbb{R}^n} \left| \det A_\xi \right|^{-p/s} \left| \varphi \left(A_\xi^{-1} (x - b) \right) \right|^p dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \left|\det A_\xi\right|^{1-p/s} \left|\varphi(y)\right|^p \, dy \\
&= \left|\det A_\xi\right|^{1-p/s} \left\|\varphi\right\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

If $p = \infty$, we get

$$
\begin{aligned}
\left\|\varphi_{s,b,\xi}\right\|_{L^\infty(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n} \left|\varphi_{s,b,\xi}(x)\right| \\
&= \sup_{x\in\mathbb{R}^n} \left|\mathrm{M}_\xi \, \mathrm{T}_{-b} \, \mathrm{D}_{s,A} \, \varphi(x)\right| \\
&= \sup_{x\in\mathbb{R}^n} \left|e^{ix\cdot\xi} \left|\det A_\xi\right|^{-1/s} \varphi\left(A_\xi^{-1}(x-b)\right)\right| \\
&= \sup_{x\in\mathbb{R}^n} \left|\left|\det A_\xi\right|^{-1/s} \varphi\left(A_\xi^{-1}(x-b)\right)\right| \\
&= \left|\det A_\xi\right|^{-1/s} \sup_{x\in\mathbb{R}^n} \left|\varphi\left(A_\xi^{-1}(x-b)\right)\right| \\
&= \left|\det A_\xi\right|^{-1/s} \left\|\varphi\right\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

Let $1 \leqslant p \leqslant \infty$. Then applying Holder's inequality, we have

$$
\begin{aligned}
\left|\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\right| &= \left|(2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\, dx\right| \\
&\leqslant (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} \left|f(x)\, e^{-ix\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\right| dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left|f(x)\right| \left|\overline{e^{ix\cdot\xi} \left|\det A_\xi\right|^{-1/s} \varphi\left(A_\xi^{-1}(x-b)\right)}\right| dx \\
&\leqslant (2\pi)^{-n/2} \left\|f\right\|_{L^{p'}(\mathbb{R}^n)} \left\|\varphi_{s,b,\xi}\right\|_{L^p(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} \left|\det A_\xi\right|^{1/p-1/s} \left\|f\right\|_{L^{p'}(\mathbb{R}^n)} \left\|\varphi\right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

We remark that if $s = p$, then Proposition 4.6 gives a weighted L^∞-estimate of the Stockwell transform. In fact,

$$
\sup_{(b,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \left|\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) \left|\det A_\xi\right|^{\frac{1}{s}-\frac{1}{p}}\right| \leqslant (2\pi)^{-n/2} \left\|f\right\|_{L^{p'}(\mathbb{R}^n)} \left\|\varphi\right\|_{L^p(\mathbb{R}^n)}.
$$

4.1 Stockwell transforms and affine Heisenberg group

Proposition 4.7. *Let $a : \mathbb{R}^n \to \mathbb{R}_+$ and $R : \mathbb{R}^n \to \mathrm{SO}(n, \mathbb{R})$ be two piecewise differentiable functions. Then we can introduce $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ as*

$$A(\xi) = a(\xi) R(\xi),$$

and write

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = (2\pi)^{-n/2} \left|a(\xi)\right|^{n/2-n/s} e^{-ib\cdot\xi} \left(f, \pi_{\mathbb{AH}}(\sigma(b,\xi))\varphi\right),$$

where $\pi_{\mathbb{AH}}$ is defined in (3.2)*, Theorem 3.26 and Theorem 3.27.*

Proof. Via direct computations, we get

$$\begin{aligned}
\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) &= \left|\det A_\xi\right|^{1/2-1/s} \mathrm{S}_{2,A,\varphi} f(b,\xi) \\
&= \left|\det(a(\xi) R(\xi))\right|^{1/2-1/s} \mathrm{S}_{2,aR,\varphi} f(b,\xi) \\
&= (2\pi)^{-n/2} \left|a(\xi)\right|^{n/2-n/s} \left(f, M_\xi T_{-b} D_{2,a(\xi)R(\xi)}\varphi\right) \\
&= (2\pi)^{-n/2} \left|a(\xi)\right|^{n/2-n/s} e^{-ib\cdot\xi} \left(f, T_{-b} M_\xi D_{2,a(\xi)R(\xi)}\varphi\right) \\
&= (2\pi)^{-n/2} \left|a(\xi)\right|^{n/2-n/s} e^{-ib\cdot\xi} \left(f, \pi_{\mathbb{AH}}(\sigma(b,\xi))\varphi\right).
\end{aligned}$$

4.2 Gabor transforms

We recall here Definition 2.2, *i.e.*, we define the **Gabor transform** $\mathrm{V}_\varphi f$ of the signal f with respect to the window φ by

$$\begin{aligned}
\left(\mathrm{V}_\varphi f\right)(b,\xi) &= (2\pi)^{-n/2} \left(f, \mathrm{M}_\xi \mathrm{T}_{-b}\varphi\right)_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \overline{\varphi((x-b))} dx
\end{aligned}$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 4.8. *We can write*

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left(\mathrm{V}_{\mathrm{D}_{s,A_\xi}\varphi} f\right)(b,\xi)$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Via direct computation, we get

$$\left(\mathrm{S}_{s,A,\varphi}\, f\right)(b,\xi) = (2\pi)^{-n/2}\left(f, \mathrm{M}_{\xi}\,\mathrm{T}_{-b}\,\mathrm{D}_{s,A_{\xi}}\,\varphi\right)_{L^2(\mathbb{R}^n)} = \left(\mathrm{V}_{\mathrm{D}_{s,A_{\xi}}\,\varphi}\, f\right)(b,\xi).$$

Proposition 4.9. *Let f be a signal in $L^2(\mathbb{R}^n)$ and let φ be a window in $L^2(\mathbb{R}^n)$. Then we can write*

$$\begin{aligned}\left(\mathrm{S}_{s,A,\varphi}\, f\right)(b,\xi) &= \left(\mathrm{D}^1_{s,A_{\xi}}\,\mathrm{D}^2_{s,\left(A_{\xi}^{-1}\right)^t}\left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\right)(b,\xi)\\ &= \left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\left(A_{\xi}^{-1}b, A_{\xi}^{t}\xi\right), \qquad b,\xi\in\mathbb{R}^n.\end{aligned}$$

Proof. Using Proposition 4.8, we get

$$\begin{aligned}&\left(\mathrm{S}_{s,A,\varphi}\, f\right)(b,\xi)\\ &= \left(\mathrm{V}_{\mathrm{D}_{s,A_{\xi}}\,\varphi}\, f\right)(b,\xi)\\ &= \left(f, \mathrm{M}_{\xi}\,\mathrm{T}_{-b}\,\mathrm{D}_{s,A_{\xi}}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\ &= \left(f, \mathrm{M}_{\xi}\,\mathrm{D}_{s,A_{\xi}}\,\mathrm{T}_{-A_{\xi}^{-1}b}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\ &= \left(f, \mathrm{D}_{s,A_{\xi}}\,\mathrm{M}_{A_{\xi}^{t}\xi}\,\mathrm{T}_{-A_{\xi}^{-1}b}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\ &= \left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f, \mathrm{M}_{A_{\xi}^{t}\xi}\,\mathrm{T}_{-A_{\xi}^{-1}b}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\ &= \left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\left(A_{\xi}^{-1}b, A_{\xi}^{t}\xi\right)\\ &= \left(\mathrm{D}^1_{s,A_{\xi}}\,\mathrm{D}^2_{s,\left(A_{\xi}^{-1}\right)^t}\left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\right)(b,\xi), \qquad b,\xi\in\mathbb{R}^n.\end{aligned}$$

Remark 4.10. Notice that, thanks to Proposition 4.9 when $A : \mathbb{R}^n \ni \xi \mapsto A(\xi) = A_{\xi} \in \mathrm{SO}(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R})$, we get

$$\begin{aligned}\left(\mathrm{S}_{s,A,\varphi}\, f\right)(b,\xi) &= \left(\mathrm{D}_{s,A_{\xi}}\left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\right)(b,\xi)\\ &= \left(\mathrm{V}_{\varphi}\left(\mathrm{D}_{s,A_{\xi}^{-1}}\, f\right)\right)\left(A_{\xi}^{-1}b, A_{\xi}^{-1}\xi\right), \qquad b,\xi\in\mathbb{R}^n.\end{aligned}$$

4.3 Moritoh wavelet transforms

We recall here Definition 2.4, *i.e.* let $R : \mathbb{R}^n \ni \xi \mapsto R(\xi) = R_\xi \in \mathrm{SO}(n,\mathbb{R})$. Then we define the **Moritoh wavelet** $\mathrm{W}_{\frac{1}{|\xi|}R^{-1},\varphi} f$ of the signal f with respect to the window φ by

$$\begin{aligned}\left(\mathrm{W}_{\frac{1}{|\xi|}R^{-1},\varphi} f\right)(b,\xi) &= |\xi|^{n/2} \int_{\mathbb{R}^n} f(x)\,\overline{\varphi\left(|\xi| R_\xi (x-b)\right)}\,dx \\ &= \left(f, \mathrm{T}_{-b}\,\mathrm{D}_{2,\frac{1}{|\xi|}R_\xi^{-1}}\,\varphi\right)_{L^2(\mathbb{R}^n)}\end{aligned}$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Notice that, setting $A_\xi = \frac{1}{|\xi|} R_\xi^{-1}$, we can write

$$\left(\mathrm{W}_{A_\xi,\varphi} f\right)(b,\xi) = \left(f, \mathrm{T}_{-b}\,\mathrm{D}_{2,A_\xi}\,\varphi\right)_{L^2(\mathbb{R}^n)}.$$

Proposition 4.11. *Let* $1 \leqslant s < \infty$, *let* $R : \mathbb{R}^n \ni \xi \mapsto R(\xi) = R_\xi \in \mathrm{SO}(n,\mathbb{R})$, *and let* $A_\xi = \frac{1}{|\xi|} R_\xi^{-1}$ *. Then we can write*

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left|\det A_\xi\right|^{1/2-1/s} (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(\mathrm{W}_{A_\xi,\mathrm{M}_{\left(A_\xi^{-1}\right)^t \xi}\varphi} f\right)(b,\xi)$$

for all $(b,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. First we can show that

$$\left(\mathrm{S}_{2,A,\varphi} f\right)(b,\xi) = (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(\mathrm{W}_{A_\xi,\mathrm{M}_{\left(A_\xi^{-1}\right)^t \xi}\varphi} f\right)(b,\xi), \qquad b,\xi \in \mathbb{R}^n.$$

In fact, via direct computation, we get for all b and ξ in $\mathbb{R}^n$,

$$\begin{aligned}&(2\pi)^{-n/2} e^{-ib\cdot\xi} \left(\mathrm{W}_{A_\xi,\mathrm{M}_{\left(A_\xi^{-1}\right)^t \xi}\varphi} f\right)(b,\xi) \\ &= (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(f, \mathrm{T}_{-b}\,\mathrm{D}_{2,A_\xi}\,\mathrm{M}_{\left(A_\xi^{-1}\right)^t \xi}\,\varphi\right) \\ &= (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(f, \mathrm{T}_{-b}\,\mathrm{M}_{A_\xi^t \left(A_\xi^{-1}\right)^t \xi}\,\mathrm{D}_{2,A_\xi}\,\varphi\right) \\ &= (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(f, \mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi\right) \\ &= (2\pi)^{-n/2} e^{-ib\cdot\xi} \left(f, e^{-ib\cdot\xi}\,\mathrm{M}_\xi\,\mathrm{T}_{-b}\,\mathrm{D}_{2,A_\xi}\,\varphi\right) \\ &= (2\pi)^{-n/2} e^{-ib\cdot\xi} e^{ib\cdot\xi} \left(f, \mathrm{M}_\xi\,\mathrm{T}_{-b}\,\mathrm{D}_{2,A_\xi}\,\varphi\right) \\ &= \left(\mathrm{S}_{2,A,\varphi} f\right)(b,\xi).\end{aligned}$$

Thanks to Remark 4.2, we get

$$\begin{aligned}
&\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) \\
&= \left|\det A_\xi\right|^{1/2-1/s} \left(\mathrm{S}_{2,A,\varphi} f\right)(b,\xi) \\
&= \left|\det A_\xi\right|^{1/2-1/s} (2\pi)^{-n/2} e^{-i b\cdot\xi} \left(\mathrm{W}_{A_\xi,\mathrm{M}_{\left(A_\xi^{-1}\right)^t \xi}} \varphi\, f\right)(b,\xi), \qquad b,\xi\in\mathbb{R}^n.
\end{aligned}$$

5 Continuous inversion formulas

Proposition 5.1. *Let f be a signal in $L^2(\mathbb{R}^n)$ and let $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1.$$

Then

$$\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\, db = \left|\det A_\xi\right|^{1-1/s} \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Proof. Thanks to the Fubini theorem and the change of variable $y = A_\xi(x-b)$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\, db \\
&= \int_{\mathbb{R}^n} \left((2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi} \overline{\varphi\left(A_\xi^{-1}(x-b)\right)} dx \right) db \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi} \overline{\varphi\left(A_\xi^{-1}(x-b)\right)} \left|\det A_\xi\right|^{-1/s} dx\, db \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \left|\det A_\xi\right|^{-1/s} \left(\int_{\mathbb{R}^n} \overline{\varphi\left(A_\xi^{-1}(x-b)\right)} db \right) dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \left|\det A_\xi\right|^{1-1/s} \left(\int_{\mathbb{R}^n} \overline{\varphi(y)} dy \right) dx \\
&= \left|\det A_\xi\right|^{1-1/s} \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n.
\end{aligned}$$

Proposition 5.2. *Let $f, \varphi \in L^2(\mathbb{R}^n)$ and $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$. Then we have*

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A_\xi}\right)(b), \qquad b,\xi \in \mathbb{R}^n,$$

where

$$f_{\xi,A_\xi}(\zeta) = \widehat{f}(\zeta) \overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}.$$

Proof. First notice that for all b and ξ in $\mathbb{R}^n$,

$$\mathrm{F}\,\mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi = \mathrm{M}_{-b}\,\mathrm{T}_{-\xi}\,\mathrm{D}_{2,\left(A_\xi^{-1}\right)^t}\,\mathrm{F}\,\varphi,$$

So, by the Plancherel formula,

$$\begin{aligned}
&\left(f, \mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\
&= \left(\mathrm{F}f, \mathrm{F}\,\mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\
&= \left(\mathrm{F}f, \mathrm{M}_{-b}\,\mathrm{T}_{-\xi}\,\mathrm{D}_{2,\left(A_\xi^{-1}\right)^t}\,\mathrm{F}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\
&= \left|\det A_\xi\right|^{1/2} \int_{\mathbb{R}^n} \widehat{f}(\zeta)\, e^{ib\cdot\zeta}\, \overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}\, d\zeta\\
&= (2\pi)^{n/2} \left|\det A_\xi\right|^{1/2} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\zeta)\, e^{ib\cdot\zeta}\, \overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}\, d\zeta\\
&= (2\pi)^{n/2} \left|\det A_\xi\right|^{1/2} \left(\mathrm{F}^{-1}_{\zeta\mapsto b}\, f_{\xi,A_\xi}\right)(b),
\end{aligned}$$

where

$$f_{\xi,A_\xi}(\zeta) = \widehat{f}(\zeta)\overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}.$$

Since we can write

$$\left(\mathrm{S}_{2,A,\varphi}\, f\right)(b,\xi) = (2\pi)^{-n/2}\, e^{-ib\cdot\xi} \left(f, \mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi\right)_{L^2(\mathbb{R}^n)}, \qquad b,\xi\in\mathbb{R}^n,$$

thanks to Remark 4.2, we get

$$\begin{aligned}
\left(\mathrm{S}_{s,A,\varphi}\, f\right)(b,\xi) &= \left|\det A_\xi\right|^{1/2-1/s} \left(\mathrm{S}_{2,A,\varphi}\, f\right)(b,\xi)\\
&= \left|\det A_\xi\right|^{1/2-1/s} (2\pi)^{-n/2}\, e^{-ib\cdot\xi} \left(f, \mathrm{T}_{-b}\,\mathrm{M}_\xi\,\mathrm{D}_{2,A_\xi}\,\varphi\right)_{L^2(\mathbb{R}^n)}\\
&= \left|\det A_\xi\right|^{1/2-1/s} (2\pi)^{-n/2}\, e^{-ib\cdot\xi}\, (2\pi)^{n/2} \left|\det A_\xi\right|^{1/2} \mathrm{F}^{-1}_{\zeta\mapsto b}\, f_{\xi,A_\xi}(b)\\
&= \left|\det A_\xi\right|^{1-1/s}\, e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b}\, f_{\xi,A_\xi}\right)(b), \qquad b,\xi\in\mathbb{R}^n.
\end{aligned}$$

5.1 Constant matrices

In this subsection we want to study matrix-valued maps $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ such that

$$A_\xi = A \in \mathrm{GL}(n,\mathbb{R}) \qquad \forall \xi \in \mathbb{R}^n, \tag{5.1}$$

and their associated Stockwell transforms.

Lemma 5.3. *Let $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ be such that*

$$A_\xi = A \in \mathrm{GL}(n,\mathbb{R}), \qquad \forall \xi \in \mathbb{R}^n,$$

and let $\varphi \in L^2(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A^t(\zeta-\xi)\right)\right|^2 |\det A|\, d\xi = \left\|\varphi\right\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. Let $\eta = A^t(\zeta - \xi)$. Then

$$\mathrm{J}_\eta(\xi) = -A^t,$$

so

$$d\eta = \left|\det \mathrm{J}_\eta(\xi)\right| d\xi = |\det A|\, d\xi.$$

Now we can write

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A^t(\zeta-\xi)\right)\right|^2 |\det A|\, d\xi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta)\right|^2 d\eta = \left\|\varphi\right\|^2_{L^2(\mathbb{R}^n)}.$$

Proposition 5.4. *Let $1 \leqslant s < \infty$, let $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ be such that*

$$A_\xi = A \in \mathrm{GL}(n,\mathbb{R}), \qquad \forall \xi \in \mathbb{R}^n,$$

and let $\varphi \in L^2(\mathbb{R}^n)$. Then

$$\begin{aligned}\left\|\varphi\right\|^2_{L^2(\mathbb{R}^n)} (f,g)_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\, \overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, |\det A|^{2/s-1}\, d\xi.\end{aligned}$$

Proof. Using Proposition 5.2, the Fubini-Tonelli theorem, Plancharel's formula and Lemma

5.3, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\,db\,|\det A|^{2/s-1}\,d\xi \\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\det A|^{1-1/s}\,e^{-ib\cdot\xi}\left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A}\right)(b) \\
&\quad\times \overline{|\det A|^{1-1/s}\,e^{-ib\cdot\xi}\,\mathrm{F}^{-1}_{\zeta\mapsto b}\, g_{\xi,A}(b)}\,db\,|\det A|^{2/s-1}\,d\xi \\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A}\right)(b)\,\overline{\left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi,A}\right)(b)}\,db\,|\det A|\,d\xi \\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{\xi,A}(\zeta)\,\overline{g_{\xi,A}(\zeta)}\,d\zeta\,|\det A|\,d\xi \\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{\varphi}\left(A^t(\zeta-\xi)\right)}\,\overline{\widehat{g}(\zeta)\overline{\widehat{\varphi}\left(A^t(\zeta-\xi)\right)}}\,d\zeta\,|\det A|\,d\xi \\
&=\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)}\int_{\mathbb{R}^n}\left|\widehat{\varphi}\left(A^t(\zeta-\xi)\right)\right|^2 |\det A|\,d\xi\,d\zeta \\
&=\|\varphi\|^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}\widehat{f}(\zeta)\overline{\widehat{g}(\zeta)}\,d\zeta \\
&=\|\varphi\|^2_{L^2(\mathbb{R}^n)}\,(f,g)_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

5.2 Diagonal matrices

In this subsection we want to study matrix-valued maps $A:\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ in the form

$$
\left(\left(A_\xi^{n\times n}\right)^{-1}\right)^t = \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \dots & \\ & & & \xi_n \end{pmatrix}, \qquad \xi_1,\dots,\xi_n \neq 0, \qquad \forall \xi \in \mathbb{R}^n, \tag{5.2}
$$

and their associated Stockwell transforms. For the sake of clarity, we write A_ξ instead of $A_\xi^{n\times n}$.

Lemma 5.5. *Let $A:\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ be such that*

$$
A_\xi^t = \begin{pmatrix} \frac{1}{\xi_1} & & & \\ & \frac{1}{\xi_2} & & \\ & & \dots & \\ & & & \frac{1}{\xi_n} \end{pmatrix}, \qquad \xi_1,\dots,\xi_n \neq 0, \qquad \forall \xi \in \mathbb{R}^n,
$$

and let $\varphi \in L^2(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)\right|^2 \left|\det A_\xi\right| d\xi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta-\underline{1})\right|^2 \left|\det A_\eta\right| d\eta,$$

where $\underline{1} = (1,1,\dots,1) \in \mathbb{R}^n$.

Proof. Notice that

$$A_\xi^t(\zeta-\xi) = A_\xi^t\zeta - \underline{1},$$

where $\underline{1} = (1,1,\dots,1) \in \mathbb{R}^n$, and let $\eta = A_\xi^t\zeta$, then

$$\begin{aligned} \mathrm{J}_\eta(\xi) &= \begin{pmatrix} -\frac{\zeta_1}{\xi_1^2} & & & \\ & -\frac{\zeta_2}{\xi_2^2} & & \\ & & \dots & \\ & & & -\frac{\zeta_n}{\xi_n^2} \end{pmatrix} = \begin{pmatrix} -\frac{\eta_1\xi_1}{\xi_1^2} & & & \\ & -\frac{\eta_2\xi_2}{\xi_2^2} & & \\ & & \dots & \\ & & & -\frac{\eta_n\xi_n}{\xi_n^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\eta_1}{\xi_1} & & & \\ & -\frac{\eta_2}{\xi_2} & & \\ & & \dots & \\ & & & -\frac{\eta_n}{\xi_n} \end{pmatrix}, \end{aligned}$$

so

$$d\eta = \left|\det \mathrm{J}_\eta(\xi)\right| d\xi = \prod_{j=1}^n \frac{\left|\eta_j\right|}{\left|\xi_j\right|} d\xi.$$

Now we can write

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)\right|^2 \frac{d\xi}{\prod_{j=1}^n \left|\xi_j\right|} = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta-\underline{1})\right|^2 \frac{d\eta}{\prod_{j=1}^n \left|\eta_j\right|}.$$

Proposition 5.6. *Let* $1 \leqslant s < \infty$, *let* $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ *be such that*

$$A_\xi^t = \begin{pmatrix} \frac{1}{\xi_1} & & & \\ & \frac{1}{\xi_2} & & \\ & & \dots & \\ & & & \frac{1}{\xi_n} \end{pmatrix}, \qquad \xi_1,\dots,\xi_n \neq 0, \qquad \forall \xi \in \mathbb{R}^n,$$

and let $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - \underline{1})\right|^2 \left|\det A_\eta\right| d\eta < \infty,$$

where $\underline{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. *Then*

$$\begin{aligned} c_\varphi (f, g)_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b, \xi) \overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b, \xi)}\, db \left|\det A_\xi\right|^{2/s-1} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b, \xi) \overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b, \xi)}\, db \frac{d\xi}{\prod_{j=1}^n \left|\xi_j\right|^{2/s-1}}. \end{aligned}$$

Proof. Using Proposition 5.2, the Fubini-Tonelli theorem, Plancharel's formula and Lemma 5.5, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b, \xi) \overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b, \xi)}\, db \left|\det A_\xi\right|^{2/s-1} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi, A_\xi}\right)(b) \\ &\quad \times \overline{\left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi, A_\xi}\right)(b)}\, db \left|\det A_\xi\right|^{2/s-1} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi, A_\xi}\right)(b) \overline{\left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi, A_\xi}\right)(b)}\, db \left|\det A_\xi\right| d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_{\xi, A_\xi}(\zeta) \overline{g_{\xi, A_\xi}(\zeta)}\, d\zeta \left|\det A_\xi\right| d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f}(\zeta) \overline{\widehat{\varphi}\left(A_\xi^t (\zeta - \xi)\right)}\, \overline{\widehat{g}(\zeta) \overline{\widehat{\varphi}\left(A_\xi^t (\zeta - \xi)\right)}}\, d\zeta \left|\det A_\xi\right| d\xi \\ &= \int_{\mathbb{R}^n} \widehat{f}(\zeta) \overline{\widehat{g}(\zeta)} \int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t (\zeta - \xi)\right)\right|^2 \left|\det A_\xi\right| d\xi\, d\zeta \\ &= \int_{\mathbb{R}^n} \widehat{f}(\zeta) \overline{\widehat{g}(\zeta)} \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - \underline{1})\right|^2 \left|\det A_\eta\right| d\eta\, d\zeta \\ &= c_\varphi \int_{\mathbb{R}^n} \widehat{f}(\zeta) \overline{\widehat{g}(\zeta)} d\zeta \\ &= c_\varphi (f, g)_{L^2(\mathbb{R}^n)}. \end{aligned}$$

5.3 Rotation matrices in dimension $n = 1, 2, 4, 8$

In this subsection we want to study matrix-valued maps $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$ for $n = 1, 2, 4, 8$.

- n=2

$$\left(\left(A_\xi^{2\times 2}\right)^{-1}\right)^t = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}; \tag{5.3}$$

- n=4

$$\left(\left(A_\xi^{4\times 4}\right)^{-1}\right)^t = \begin{pmatrix} \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 \\ \xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ \xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ \xi_4 & -\xi_3 & \xi_2 & \xi_1 \end{pmatrix}; \tag{5.4}$$

- n=8

$$\begin{aligned} &\left(\left(A_\xi^{8\times 8}\right)^{-1}\right)^t \\ &= \begin{pmatrix} \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6 & -\xi_7 & -\xi_8 \\ \xi_2 & \xi_1 & -\xi_4 & \xi_3 & -\xi_6 & \xi_5 & \xi_8 & -\xi_7 \\ \xi_3 & \xi_4 & \xi_1 & -\xi_2 & -\xi_7 & -\xi_8 & \xi_5 & \xi_6 \\ \xi_4 & -\xi_3 & \xi_2 & \xi_1 & -\xi_8 & \xi_7 & -\xi_6 & \xi_5 \\ \xi_5 & \xi_6 & \xi_7 & \xi_8 & \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 \\ \xi_6 & -\xi_5 & \xi_8 & -\xi_7 & \xi_2 & \xi_1 & \xi_4 & -\xi_3 \\ \xi_7 & -\xi_8 & -\xi_5 & \xi_6 & \xi_3 & -\xi_4 & \xi_1 & \xi_2 \\ \xi_8 & \xi_7 & -\xi_6 & -\xi_5 & \xi_4 & \xi_3 & -\xi_2 & \xi_1 \end{pmatrix}, \end{aligned} \tag{5.5}$$

  and their associated Stockwell transforms.

Lemma 5.7. *Let $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$ be a matrix-valued function such that*

(a) *$\frac{1}{|\xi|} A_\xi^{-1} \in \mathrm{SO}(n, \mathbb{R})$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$;*

(b) *there exists a matrix $P \in \mathrm{O}(n, \mathbb{R})$ such that $A_\xi^{-1} \zeta = P A_\zeta^{-1} \xi$;*

and let $\eta = A_\xi^t \zeta$.

Then

$$\left|\det \mathrm{J}_\eta(\xi)\right| = \frac{\left|\det A_\eta^{-1}\right|}{\left|\det A_\xi^{-1}\right|} = \frac{|\eta|^n}{|\xi|^n}.$$

Proof. Using *(a)*, we get

$$\eta = A_\xi^t \zeta = \left(\left(|\xi| \frac{1}{|\xi|} A_\xi^{-1}\right)^{-1}\right)^t \zeta = \left(\frac{1}{|\xi|}\left(\frac{1}{|\xi|} A_\xi^{-1}\right)^t\right)^t \zeta = \frac{1}{|\xi|^2} A_\xi^{-1} \zeta.$$

Then, thanks to *(b)*, we have

$$\eta = A_\xi^t \zeta = \frac{1}{|\xi|^2} A_\xi^{-1} \zeta = \frac{1}{|\xi|^2} P A_\zeta^{-1} \xi = P A_\zeta^{-1} \frac{\xi}{|\xi|^2},$$

and

$$\begin{aligned} J_\eta(\xi) &= P A_\zeta^{-1} \begin{pmatrix} \frac{|\xi|^2 - 2\xi_1^2}{|\xi|^4} & \frac{-2\xi_1\xi_2}{|\xi|^4} & \dots & \frac{-2\xi_1\xi_n}{|\xi|^4} \\ \frac{-2\xi_2\xi_1}{|\xi|^4} & \frac{|\xi|^2 - 2\xi_2^2}{|\xi|^4} & \dots & \frac{-2\xi_2\xi_n}{|\xi|^4} \\ \dots & \dots & \dots & \dots \\ \frac{-2\xi_n\xi_1}{|\xi|^4} & \frac{-2\xi_n\xi_2}{|\xi|^4} & \dots & \frac{|\xi|^2 - 2\xi_n^2}{|\xi|^4} \end{pmatrix} \\ &= \frac{1}{|\xi|^4} P A_\zeta^{-1} \left(|\xi|^2 \,\mathrm{Id} - 2 C_\xi \right) \\ &= \frac{1}{|\xi|^2} P A_\zeta^{-1} \left(\mathrm{Id} - \frac{2}{|\xi|^2} C_\xi \right), \end{aligned}$$

where

$$C_\xi = \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \dots & \xi_1\xi_n \\ \xi_2\xi_1 & \xi_2^2 & \cdots & \xi_2\xi_n \\ \dots & \dots & \dots & \dots \\ \xi_n\xi_1 & \xi_n\xi_2 & \cdots & \xi_n^2 \end{pmatrix},$$

and Id is the n-dimensional identity matrix.

At this point it is useful to prove that

$$\det \left(\mathrm{Id} - \frac{2}{|\xi|^2} C_\xi \right) = -1.$$

To do this first observe that rank $C_\xi = 1$, C_ξ is a symmetrical matrix, so there exists only one non-zero eigenvalue. We can check that $|\xi|^2$ is the only non-zero eigenvalue associated to the eigenvector $(\xi_1, \xi_2, \dots, \xi_n)$, in fact

$$\begin{aligned} C_\xi \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} &= \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \dots & \xi_1\xi_n \\ \xi_2\xi_1 & \xi_2^2 & \cdots & \xi_2\xi_n \\ \dots & \dots & \dots & \dots \\ \xi_n\xi_1 & \xi_n\xi_2 & \cdots & \xi_n^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 & 0 & \dots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & \xi_n \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \dots & \dots & \dots & \dots \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} \end{aligned}$$

$$= |\xi|^2 \begin{pmatrix} \xi_1 & 0 & \dots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & \xi_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = |\xi|^2 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix}.$$

So the eigenvalues $\lambda_1^C, \dots, \lambda_n^C$ of $-\frac{2}{|\xi|^2} C_\xi$ are $\lambda_1^C = \cdots = \lambda_{n-1}^C = 0$ and $\lambda_n^C = -2$.

We introduce

$$B_\xi = \mathrm{Id} - \frac{2}{|\xi|^2} C_\xi,$$

and we compute the eigenvalues λ^B of B_ξ by studying its characteristic equation, that is,

$$\begin{aligned} 0 &= \det\left(\lambda^B \,\mathrm{Id} - B_\xi\right) \\ &= \det\left(\lambda^B \,\mathrm{Id} - \mathrm{Id} + \frac{2}{|\xi|^2} C_\xi\right) \\ &= \det\left((\lambda^B - 1)\,\mathrm{Id} - \left(-\frac{2}{|\xi|^2} C_\xi\right)\right). \end{aligned}$$

From this follows that the eigenvalues of B_ξ are given by

$$\lambda_j^B = \lambda_j^C + 1, \qquad j = 1, \dots, n.$$

So $\lambda_1^B = \cdots = \lambda_{n-1}^B = 1$ and $\lambda_n^B = -1$ and it is sufficient to observe that

$$\det\left(\mathrm{Id} - \frac{2}{|\xi|} C_\xi\right) = \det B_\xi = \prod_{j=1}^n \lambda_j^B = -1.$$

By the preceding observations,

$$\begin{aligned} \det J_\eta(\xi) &= \det\left(\frac{1}{|\xi|^2} P A_\zeta^{-1} \left(\mathrm{Id} - \frac{2}{|\xi|^2} C_\xi\right)\right) \\ &= \det\left(\frac{1}{|\xi|^2} P A_\zeta^{-1}\right) \det\left(\mathrm{Id} - \frac{2}{|\xi|^2} C_\xi\right) \\ &= -\frac{1}{|\xi|^{2n}} \det\left(P A_\zeta^{-1}\right) \\ &= -\frac{1}{|\xi|^{2n}} (\det P) \left(\det A_\zeta^{-1}\right). \end{aligned}$$

It is useful to point out that

$$\det A_\zeta^{-1} = \det\left(|\zeta| \frac{1}{|\zeta|} A_\zeta^{-1}\right) = |\zeta|^n \det\left(\frac{1}{|\zeta|} A_\zeta^{-1}\right) = |\zeta|^n ,$$

and, thanks to the fact that $\left|\eta\right| = \frac{1}{|\xi|} |\zeta|$, we get

$$\det A_\zeta^{-1} = |\zeta|^n = \left|\eta\right|^n |\xi|^n .$$

In the end we get

$$\left|\det \mathrm{J}_\eta(\xi)\right| = \frac{1}{|\xi|^{2n}} |\det P| \left|\det A_\zeta^{-1}\right| = \frac{1}{|\xi|^{2n}} \left|\eta\right|^\eta |\xi|^\eta = \frac{\left|\eta\right|^\eta}{|\xi|^n}.$$

Lemma 5.8. *Let $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$ be such that*

(a) $\frac{1}{|\xi|} A_\xi^{-1} \in \mathrm{SO}(n, \mathbb{R})$ *for every* $\xi \in \mathbb{R}^n \setminus \{0\}$;

(b) *there exists a matrix* $P \in \mathrm{O}(n, \mathbb{R})$ *such that* $A_\xi^{-1}\zeta = P A_\zeta^{-1}\xi$;

(c) *for every* $\xi \in \mathbb{R}^n$ *we have*

$$A_\xi^t \xi = e_1,$$

where $e_1 = (1, 0 \ldots, 0) \in \mathbb{R}^n$.

Then we have

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta - \xi)\right)\right|^2 \frac{d\xi}{|\xi|^n} = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - e_1)\right|^2 \frac{d\eta}{\left|\eta\right|^n}.$$

Proof. Let $\eta = A_\xi^t \zeta$, then using Lemma 5.7 we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta - \xi)\right)\right|^2 \frac{d\xi}{|\xi|^n} \\
&= \int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t \zeta - A_\xi^t \xi\right)\right|^2 \frac{d\xi}{|\xi|^n} \\
&= \int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t \zeta - e_1\right)\right|^2 \frac{d\xi}{|\xi|^n} \\
&= \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - e_1)\right|^2 \frac{d\eta}{\left|\eta\right|^n}.
\end{aligned}$$

Proposition 5.9. *Let $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$ be such that*

(a) $\frac{1}{|\xi|} A_\xi^{-1} \in \mathrm{SO}(n, \mathbb{R})$ *for every* $\xi \in \mathbb{R}^n \setminus \{0\}$;

(b) *there exists a matrix* $P \in \mathrm{O}(n,\mathbb{R})$ *such that* $A_\xi^{-1}\zeta = PA_\zeta^{-1}\xi$;

(c) *for every* $\xi \in \mathbb{R}^n$ *we have*

$$A_\xi^t \xi = |\xi|\, e_1,$$

where $e_1 = (1,0\ldots,0) \in \mathbb{R}^n$.

Furthermore let $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - e_1)\right|^2 \frac{d\eta}{|\eta|^n} < \infty,$$

then we have

$$\begin{aligned}
c_\varphi (f,g)_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, \frac{d\xi}{|\xi|^{n(2/s-1)}}.
\end{aligned}$$

Proof. Using Proposition 5.2, the Fubini-Tonelli theorem, Plancharel's formula and Lemma 5.8, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A_\xi}\right)(b) \\
&\times \overline{\left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi,A_\xi}\right)(b)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A_\xi}\right)(b)\,\overline{\left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi,A_\xi}\right)(b)}\, db\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{\xi,A_\xi}(\zeta)\,\overline{g_{\xi,A_\xi}(\zeta)}\, d\zeta\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}\,\overline{\widehat{g}(\zeta)\overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}}\, d\zeta\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} \int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)\right|^2 \frac{d\xi}{|\xi|^n}\, d\zeta \\
=&\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta - e_1)\right|^2 \frac{d\eta}{|\eta|^n}\, d\zeta \\
=&c_\varphi \int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} d\zeta \\
=&c_\varphi (f,g)_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

It is worth to mention that, under this set of hypotheses, it is not possible to extend the result given in Proposition 5.9 to any dimension n. In fact we have the following result.

Proposition 5.10. *Let $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$, then there does not exist a continuous mapping $A : S^{n-1} \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$, such that for every $\xi \in S^{n-1}$ the vector $A_\xi \xi$ is parallel to e_1.*

Proof. The proof of Proposition 5.10 can be found in [28] and it is based on [5] by Bott and Milnor.

Remark 5.11. In view of Proposition 5.10, Hypothesis *(c) of Lemma 5.8 and Proposition 5.9 cannot be satisfied for any $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$.*

Remark 5.12. Matrix-valued maps as in (1.1), (1.2) and (1.4) satisfy hypothesys *(a)*,*(b)* and *(c)* of Lemma *5.8* and Proposition *5.9*. We can easily check this fact for $A_\xi^{2\times 2}$. In this case,

$$\frac{1}{|\xi|}\left(A_\xi^{2\times 2}\right)^{-1} = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix} \in \mathrm{SO}(n, \mathbb{R}),$$

so $A_\xi^{2\times 2}$ satisfies *(a)*.

Furthermore, taking

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{O}(n, \mathbb{R}),$$

we get

$$\begin{aligned}
\left(A_\xi^{2\times 2}\right)^{-1}\zeta &= \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \\
&= \begin{pmatrix} \xi_1\zeta_1 + \xi_2\zeta_2 \\ -\xi_2\zeta_1 + \xi_1\zeta_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \zeta_1 & \zeta_2 \\ -\zeta_2 & \zeta_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \\
&= P\left(A_\zeta^{2\times 2}\right)^{-1}\xi,
\end{aligned}$$

that is *(b)*.

Finally

$$\left(A_\xi^{2\times 2}\right)^t = \left(\left(\left(A_\xi^{2\times 2}\right)^{-1}\right)^t\right)^{-1} = \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix},$$

and

$$\left(A_\xi^{2\times 2}\right)^t \xi = \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1,$$

that gives us *(c)*.

5.4 A unification

Lemma 5.13. *Let* $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ *be a piecewise differentiable function such that we can find a fixed* $(1,2)$*-tensor* F *and a fixed* $(1,1)$*-tensor* G *such that*

$$\left(\left(A_\xi^t\right)^{-1}\right)_j^i = F_{jl}^i \xi^l + G_j^i.$$

Then

$$\int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)\right|^2 \left|\det A_\xi\right| d\xi = \int_{\eta_\zeta(\mathbb{R}^n)} \left|\widehat{\varphi}(\eta)\right|^2 \frac{d\eta}{\left|\det\left(F_{jk}^i \eta^j + \delta_k^i\right)\right|}.$$

where $\eta_\zeta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is given by*

$$\eta_\zeta(\xi) = A_\xi^t(\zeta-\xi).$$

Proof. The proof of this lemma is an extension of the one given in [21]. Introduce for simplicity the $(1,1)$-tensor

$$Q_i^j = \left(A_\xi^t\right)_i^j,$$

and its inverse

$$S_j^i = \left(\left(A_\xi^t\right)^{-1}\right)_j^i,$$

then we can define $\eta : \mathbb{R}^n \ni \zeta \mapsto \eta_\zeta(\xi) \in \mathbb{R}^n$, where

$$\left(\eta_\zeta(\xi)\right)^j = Q_i^j (\zeta-\xi)^i.$$

Then

$$\left(J_\eta(\xi)\right)_k^j = \partial_k Q_i^j (\zeta-\xi)^i + Q_i^j \partial_k (\zeta-\xi)^i$$

$$
\begin{aligned}
&= \partial_k Q_i^j (\zeta - \xi)^i - Q_i^j \delta_k^i \\
&= Q_i^j \left(S_j^i \partial_k Q_i^j (\zeta - \xi)^i - \delta_k^i \right) \\
&= Q_i^j \left(- \left(\partial_k S_j^i \right) Q_i^j (\zeta - \xi)^i - \delta_k^i \right) \\
&= Q_i^j \left(- \left(\partial_k S_j^i \right) \eta^j - \delta_k^i \right).
\end{aligned}
$$

Using the hypothesis, we know that

$$
S_j^i = \left(\left(A_\xi^{-1} \right)^t \right)_j^i = F_{jl}^i \xi^l + G_j^i,
$$

so

$$
\partial_k S_j^i = F_{jl}^i \delta_k^l = F_{jk}^i,
$$

and

$$
\left(J_\eta (\xi) \right)_k^j = Q_i^j \left(-F_{jk}^i \eta^j - \delta_k^i \right).
$$

Observe that

$$
\det Q_i^j = \det A_\xi^t = \det A_\xi,
$$

and

$$
\det \left(J_\eta (\xi) \right)_k^j = (-1)^n \left(\det A_\xi \right) \det \left(F_{jk}^i \eta^j + \delta_k^i \right),
$$

so

$$
d\eta = \left| \det \left(J_\eta (\xi) \right)_k^j \right| d\xi = \left| \det A_\xi \right| \left| \det \left(F_{jk}^i \eta^j + \delta_k^i \right) \right|.
$$

In the end, we have

$$
\int_{\mathbb{R}^n} \left| \widehat{\varphi} \left(A_\xi^t (\zeta - \xi) \right) \right|^2 \left| \det A_\xi \right| d\xi = \int_{\eta_\zeta(\mathbb{R}^n)} \left| \widehat{\varphi} (\eta) \right|^2 \frac{d\eta}{\left| \det \left(F_{jk}^i \eta^j + \delta_k^i \right) \right|}.
$$

Theorem 5.14. *Let $1 \leqslant s < \infty$ and let $A : \mathbb{R}^n \ni \omega \mapsto A(\omega) = A_\xi \in \mathrm{GL}(n, \mathbb{R})$ be a piecewise differentiable function such that we can find a fixed $(1,2)$-tensor F and a fixed $(1,1)$-tensor*

G such that

$$\left(\left(A_\xi^{-1}\right)^t\right)_j^i = F_{jl}^i \xi^l + G_j^i,$$

and the function

$$\begin{aligned} \eta_\zeta : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \xi &\longmapsto A_\xi^t (\zeta - \xi), \end{aligned}$$

is such that $\eta_\zeta(\mathbb{R}^n) = \mathbb{R}^n$ *for all* $\zeta \in \mathbb{R}^n$.

Let $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} \left|\widehat{\varphi}(\eta)\right|^2 \frac{d\eta}{\left|\det\left(F_{jk}^i \eta^j + \delta_k^i\right)\right|} < \infty.$$

Then

$$c_\varphi (f,g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi.$$

Proof. Using Proposition 5.2, the Fubini-Tonelli theorem, Plancharel's formula and Lemma 5.13 we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} g\right)(b,\xi)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi}\left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A_\xi}\right)(b) \\
&\times \overline{\left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi}\left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi,A_\xi}\right)(b)}\, db\, \left|\det A_\xi\right|^{2/s-1} d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathrm{F}^{-1}_{\zeta\mapsto b} f_{\xi,A_\xi}\right)(b)\,\overline{\left(\mathrm{F}^{-1}_{\zeta\mapsto b} g_{\xi,A_\xi}\right)(b)}\, db\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{\xi,A_\xi}(\zeta)\,\overline{g_{\xi,A_\xi}(\zeta)}\, d\zeta\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}\,\overline{\widehat{g}(\zeta)\overline{\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)}}\, d\zeta\, \left|\det A_\xi\right| d\xi \\
=&\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} \int_{\mathbb{R}^n} \left|\widehat{\varphi}\left(A_\xi^t(\zeta-\xi)\right)\right|^2 \left|\det A_\xi\right| d\xi\, d\zeta \\
=&\int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} \int_{\eta_\zeta(\mathbb{R}^n)} \left|\widehat{\varphi}(\eta)\right|^2 \frac{d\eta}{\left|\det\left(F_{jk}^i \eta^j + \delta_k^i\right)\right|}\, d\zeta \\
=&c_\varphi \int_{\mathbb{R}^n} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)} d\zeta
\end{aligned}$$

$$=c_{\varphi}\left(f,g\right)_{L^2(\mathbb{R}^n)}.$$

Remark 5.15. We can recover Proposition 5.4, Proposition 5.6 and Proposition 5.9 as corollaries of Theorem 5.14 .

6 Localization operators

We first introduce localization operators associated to the strictly-admissible section studied in Theorem 3.27, *i.e.*, for all f_1 and f_2 in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}&\left(L_{F,\sigma,H,\varphi} f_1, f_2\right)\\&\quad = \frac{1}{c_{\sigma,H,\varphi}} \int_{\mathbb{AH}/H} F(b,\xi)\left(f_1, \pi_{\mathbb{AH}}(\sigma((b,\xi)))\varphi\right)\left(\pi_{\mathbb{AH}}(\sigma((b,\xi)))\varphi, f_2\right) db\, d\xi.\end{aligned}$$

Then we can define and study localization operators associated to multi-dimensional Stockwell transforms using Proposition 4.7.

Definition 6.1. Let $a : \mathbb{R}^n \to \mathbb{R}_+$ and $R : \mathbb{R}^n \to \mathrm{SO}(n,\mathbb{R})$ be two piecewise differentiable functions. Then we can introduce $A : \mathbb{R}^n \to \mathrm{GL}(n,\mathbb{R})$ as

$$A(\xi) = a(\xi) R(\xi), \qquad \xi \in \mathbb{R}^n,$$

and write

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = (2\pi)^{-n/2}\left|a(\xi)\right|^{n/2-n/s} e^{-ib\cdot\xi}\left(f, \pi_{\mathbb{AH}}(\sigma(b,\xi))\varphi\right), \qquad b,\xi \in \mathbb{R}^n,$$

where $\pi_{\mathbb{AH}}$ is defined in (3.2), Theorem 3.26 and Theorem 3.27. Now assume that A and φ are such that they satisfy the assumptions of Theorem 5.14, then the localization operator $L^s_{F,A,\varphi}$ associated to the multi-dimensional Stockwell transform and with symbol $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\begin{aligned}&\left(L^s_{F,A,\varphi} f_1, f_2\right)\\&\quad = \frac{1}{c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(b,\xi)\left(\mathrm{S}_{s,A,\varphi} f_1\right)(b,\xi)\,\overline{\left(\mathrm{S}_{s,A,\varphi} f_2\right)(b,\xi)}\, db\,\left|\det A_\xi\right|^{2/s-1} d\xi\end{aligned}$$

for all f_1 and f_2 in $L^2(\mathbb{R}^n)$.

Notice that these localization operators extend the ones given in [20].

Proposition. *The localization operator* $L^2_{F,A,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and

$$\left\|L^2_{F,A,\varphi}\right\|_{B(L^2(\mathbb{R}^n))} \leqslant \frac{1}{c_\varphi}\|F\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Furthermore, it is in the trace class S_1, and its trace is given by

$$\operatorname{tr}\left(L^2_{F,\varphi}\right) = \frac{1}{c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\,db\,d\xi.$$

Proof. It follows directly from Proposition 4.7, Definition 6.1, Proposition 3.3 and Proposition 3.4.

We can elucidate the link between localization operators associated to the multi-dimensional Stockwell transform and the Weyl transforms.

Proposition 6.2. *Let $\varphi \in L^2(\mathbb{R}^n)$ be an admissible function as in Theorem 5.14 such that $\left\|\varphi\right\|_{L^2(\mathbb{R}^n)} = 1$. Then, for $1 \leqslant s < \infty$ and for all $F \in L^1(\mathbb{R}^n\times\mathbb{R}^n) \cap L^2(\mathbb{R}^n\times\mathbb{R}^n)$,*

$$L^s_{F,A,\varphi} = W_\sigma,$$

where

$$\sigma(q,p) = (2\pi)^{-n/2}\frac{1}{c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(b,\xi)\,\overline{W(\varphi,\varphi)\left(A_\xi^{-1}(q-b),A_\xi^{t}(p-\xi)\right)}\,db\,d\xi,$$

for all $(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$.

Proof. The proof given here follows, with minor changes, the one given in [20] for the 1-dimensional Stockwell transform. First notice that for all $q,p\in\mathbb{R}^n$,

$$\begin{aligned}
&\left(\operatorname{Wig}\left(\varphi_{s,b,\xi},\varphi_{s,b,\xi}\right)\right)(q,p)\\
&=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot p}\varphi_{s,b,\xi}\left(q+\frac{x}{2}\right)\overline{\varphi_{s,b,\xi}\left(q-\frac{x}{2}\right)}\,dx\\
&=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot p}\left(\mathrm{M}_\xi\,\mathrm{T}_{-b}\,\mathrm{D}_{s,A_\xi}\varphi\right)\left(q+\frac{x}{2}\right)\overline{\left(\mathrm{M}_\xi\,\mathrm{T}_{-b}\,\mathrm{D}_{s,A_\xi}\varphi\right)\left(q-\frac{x}{2}\right)}\,dx\\
&=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot(p-\xi)}\varphi\left(A_\xi^{-1}\left(q-b+\frac{x}{2}\right)\right)\\
&\quad\times\overline{\varphi\left(A_\xi^{-1}\left(q-b-\frac{x}{2}\right)\right)}\left|\det A_\xi\right|^{-2/s}dx\\
&=\left|\det A_\xi\right|^{1-2/s}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot A_\xi^t(p-\xi)}\varphi\left(A_\xi^{-1}(q-b)+\left(\frac{y}{2}\right)\right)\\
&\quad\times\overline{\varphi\left(A_\xi^{-1}(q-b)-\left(\frac{y}{2}\right)\right)}\,dy
\end{aligned}$$

$$=\left|\det A_{\xi}\right|^{1-2/s}(\mathrm{Wig}(\varphi,\varphi))\left(A_{\xi}^{-1}(q-b),A_{\xi}^{t}(p-\xi)\right).$$

Using the Moyal identity to the effect that for u_1, u_2, v_1 and v_2 in $L^2(\mathbb{R}^n)$

$$(\mathrm{Wig}(u_1,v_1),\mathrm{Wig}(u_2,v_2))_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}=(u_1,u_2)_{L^2(\mathbb{R}^n)}\overline{(v_1,v_2)_{L^2(\mathbb{R}^n)}},$$

we can write

$$\begin{aligned}
&\left(L^s_{F,A,\varphi}f_1,f_2\right)\\
=&\frac{1}{c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\left(\mathrm{S}_{s,A,\varphi}f_1\right)(b,\xi)\overline{\left(\mathrm{S}_{s,A,\varphi}f_2\right)(b,\xi)}\,db\left|\det A_\xi\right|^{2/s-1}d\xi\\
=&\frac{1}{(2\pi)^n c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\left(f_1,\varphi_{s,b,\xi}\right)_{L^2(\mathbb{R}^n)}\overline{\left(f_2,\varphi_{s,b,\xi}\right)_{L^2(\mathbb{R}^n)}}\,db\left|\det A_\xi\right|^{2/s-1}d\xi\\
=&\frac{1}{(2\pi)^n c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\left(\mathrm{Wig}(f_1,f_2),\mathrm{Wig}\left(\varphi_{s,b,\xi},\varphi_{s,b,\xi}\right)\right)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}\\
&\times db\left|\det A_\xi\right|^{2/s-1}d\xi.
\end{aligned}$$

Thanks to the preceding computations, we know that

$$\begin{aligned}
&\left(\mathrm{Wig}(f_1,f_2),\mathrm{Wig}\left(\varphi_{s,b,\xi},\varphi_{s,b,\xi}\right)\right)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}\\
=&\left|\det A_\xi\right|^{1-2/s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathrm{Wig}(f_1,f_2))(q,p)\\
&\times(\mathrm{Wig}(\varphi,\varphi))\left(A_\xi^{-1}(q-b),A_\xi^t(p-\xi)\right)dq\,dp,
\end{aligned}$$

so, thanks to Fubini's Theorem,

$$\begin{aligned}
&\left(L^s_{F,A,\varphi}f_1,f_2\right)\\
=&\frac{1}{(2\pi)^n c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\left(\mathrm{Wig}(f_1,f_2),\mathrm{Wig}\left(\varphi_{s,b,\xi},\varphi_{s,b,\xi}\right)\right)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}\\
&\times db\left|\det A_\xi\right|^{2/s-1}d\xi\\
=&\frac{1}{(2\pi)^n c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\\
&\times\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathrm{Wig}(f_1,f_2))(q,p)(\mathrm{Wig}(\varphi,\varphi))\left(A_\xi^{-1}(q-b),A_\xi^t(p-\xi)\right)dq\,dp\right)db\,d\xi\\
=&(2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{1}{(2\pi)^{n/2}c_\varphi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(b,\xi)\right.\\
&\left.\times(\mathrm{Wig}(\varphi,\varphi))\left(A_\xi^{-1}(q-b),A_\xi^t(p-\xi)\right)d\xi\,db\right)(\mathrm{Wig}(f_1,f_2))(q,p)\,dq\,dp
\end{aligned}$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(q,p) \left(\mathrm{Wig}(f_1, f_2)\right)(q,p)\, dq\, dp$$
$$= (\mathrm{W}_\sigma f_1, f_2)_{L^2(\mathbb{R}^n)}.$$

It is possible to study localization operators in a slightly more general fashion. In fact, we can define $L^s_{F,A,\varphi}$ by

$$\left(L^s_{F,A,\varphi} f_1, f_2\right) = \frac{1}{c_\varphi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(b,\xi) \left(\mathrm{S}_{s,A,\varphi} f_1\right)(b,\xi) \overline{\left(\mathrm{S}_{s,A,\varphi} f_2\right)(b,\xi)}\, db \left|\det A_\xi\right|^{2/s-1} d\xi$$

for all f_1 and f_2 in $L^2(\mathbb{R}^n)$, where $A : \mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$ is such that it satisfies the assumptions of Theorem 5.14. In this case we have to drop the connections with the affine Heisenberg group, but still Proposition 6.2 holds with no modifications.

7 Instantaneous frequency

7.1 Phase of the multi-dimensional Stockwell transform and instantaneous frequency

Let $f \in L^2(\mathbb{R}^n)$ be a complex-valued signal. Then we can write f in the polar form

$$f(x) = a(x)\, e^{i\vartheta(x)}, \qquad x \in \mathbb{R}^n,$$

where $a(x) = \left|f(x)\right|$ and we assume that $a(x) \neq 0$ for all $x \in \mathbb{R}^n$. The phase of the signal allow us to define the instantaneous frequency IFf by

$$(IFf)(x) = (\nabla\vartheta)(x), \qquad x \in \mathbb{R}^n.$$

Proposition 7.1. *Let $f \in L^2(\mathbb{R}^n)$ be a complex-valued signal. Then we can find $\gamma : \mathbb{R}^n \to \mathbb{C}$, $r_1 : \mathbb{R}^n \to \mathbb{C}$ and $r_2 : \mathbb{R}^n \to \mathbb{C}$ with $r_1, r_2 \in \mathcal{O}\left(|x|^2\right)$, i.e., there exist $M_1, M_2, \delta_1, \delta_2 > 0$ such that*

$$\left|r_j(x)\right| \leqslant M_j |x|^2, \qquad \forall x \in B_{\delta_j}(0), \quad j = 1, 2, \tag{7.1}$$

where

$$B_{\delta_j}(0) = \left\{x \in \mathbb{R}^n : |x| < \delta_j\right\}.$$

Then

$$\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} f(b) \left[\overline{\widehat{\varphi}\left(A_\xi^t\left((IFf)(b) - \xi\right)\right)} + \varepsilon(b,\xi)\right],$$

where

$$\varepsilon(b,\xi) = \varepsilon_1(b,\xi) + \varepsilon_2(b,\xi) + \varepsilon_3(b,\xi),$$

and

$$\varepsilon_1(b,\xi) = (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{a(b)} \int_{\mathbb{R}^n} e^{-ix\cdot(\xi - (\nabla\vartheta)(b))} (\nabla a)(b) \cdot x\, \overline{\varphi\left(A_\xi^{-1} x\right)}\, dx,$$

$$\varepsilon_2(b,\xi) = (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{a(b)} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{\vartheta(x+b)-\vartheta(b)}\, r_1(x)\, \overline{\varphi\left(A_\xi^{-1}x\right)}\, dx,$$

$$\begin{aligned}\varepsilon_3(b,\xi) &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x)\; r_2(x)\, \overline{\varphi\left(A_\xi^{-1}x\right)}\, dx \\ &\quad + (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{a(b)} \int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x) \\ &\quad \times r_2(x)(\nabla a)(b)\cdot x\, \overline{\varphi\left(A_\xi^{-1}x\right)}\, dx.\end{aligned}$$

Proof. Let $A:\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n,\mathbb{R})$. Then we can write for all b and ξ in $\mathbb{R}^n$,

$$\begin{aligned}\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\, dx \\ &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} a(x)\, e^{i\vartheta(x)} e^{-ix\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}(x-b)\right)}\, dx \\ &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} a(x+b)\, e^{i\vartheta(x+b)} e^{-i(x+b)\cdot\xi}\, \overline{\varphi\left(A_\xi^{-1}x\right)}\, dx \\ &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} a(x+b)\, e^{-i((x+b)\cdot\xi-\vartheta(x+b))} \\ &\quad \times \overline{\varphi\left(A_\xi^{-1}x\right)}\, dx.\end{aligned}$$

By Taylor's formula,

$$a(x+b) = a(b) + (\nabla a)(b)\cdot x + r_1(x),$$

and

$$\vartheta(x+b) = \vartheta(b) + (\nabla\vartheta)(b)\cdot x + r_2(x),$$

with r_1, $r_2 \in \mathcal{O}\left(|x|^2\right)$, *i.e.*, there exist M_1, M_2, δ_1, $\delta_2 > 0$ such that

$$\left|r_j(x)\right| \leqslant M_j\, |x|^2, \qquad \forall x \in B_{\delta_j}(0), \quad j = 1,2,$$

where

$$B_{\delta_j}(0) = \left\{x \in \mathbb{R}^n : |x| < \delta_j\right\}.$$

So, for all b and ξ in $\mathbb{R}^n$,

$$\begin{aligned}&e^{i(b\cdot\xi-\vartheta(b))}\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) \\ &= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1/s} \int_{\mathbb{R}^n} \left(a(b) + (\nabla a)(b)\cdot x + r_1(x)\right)\end{aligned}$$

$$
\begin{aligned}
&\times e^{-ix\cdot(\xi-\nabla\vartheta(b))}e^{ir_2(x)}\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}e^{ir_2(x)}a(b)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}e^{ir_2(x)}\,\nabla a(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}e^{ir_2(x)}\,r_1(x)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx.
\end{aligned}
$$

Notice that there exists $\gamma:\mathbb{R}^n\to\mathbb{R}$ such that $\left|\gamma(x)\right|=1$ for all $x\in\mathbb{R}^n$ and

$$
e^{ir_2(x)}=1+\gamma(x)\,r_2(x).
$$

With the change of variable $y=A_\xi^{-1}x$, we get for all b and ξ in $\mathbb{R}^n$,

$$
\begin{aligned}
&(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}a(b)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{1-1/s}\int_{\mathbb{R}^n}e^{-iA_\xi y\cdot(\xi-(\nabla\vartheta)(b))}a(b)\,\overline{\varphi(y)}dy\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{1-1/s}a(b)\int_{\mathbb{R}^n}e^{-iy\cdot A_\xi^t(\xi-(\nabla\vartheta)(b))}\,\overline{\varphi(y)}dy\\
&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{1-1/s}a(b)\overline{\int_{\mathbb{R}^n}e^{-iy\cdot A_\xi^t((\nabla\vartheta)(b)-\xi)}\,\varphi(y)\,dy}\\
&=\left|\det A_\xi\right|^{1-1/s}a(b)\,\overline{\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)}.
\end{aligned}
$$

It follows that for all b and ξ in $\mathbb{R}^n$,

$$
\begin{aligned}
&e^{i(b\cdot\xi-\vartheta(b))}\left(\mathrm{S}_{s,A,\varphi}f\right)(b,\xi)\\
&=\left|\det A_\xi\right|^{1-1/s}a(b)\,\overline{\widehat{\varphi}\left(A_\xi^t\left(\nabla\vartheta(b)-\xi\right)\right)}\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-\nabla\vartheta(b))}\gamma(x)\,r_2(x)a(b)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-\nabla\vartheta(b))}e^{ir_2(x)}\,\nabla a(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi-\nabla\vartheta(b))}e^{ir_2(x)}\,r_1(x)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx.
\end{aligned}
$$

Using the preceding observations

$$
\begin{aligned}
&e^{i(b\cdot\xi-\vartheta(b))}\left(\mathrm{S}_{s,A,\varphi}f\right)(b,\xi)\\
&=\left|\det A_\xi\right|^{1-1/s}a(b)\,\overline{\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x)\, r_2(x) a(b)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\,\nabla a(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x)\, r_2(x)(\nabla a)(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1/s}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{i(\vartheta(x+b)-\vartheta(b))}\, r_1(x)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx,\qquad b,\xi\in\mathbb{R}^n.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&e^{i(b\cdot\xi-\vartheta(b))}\left(\mathrm{S}_{s,A,\varphi}f\right)(b,\xi)\\
&\qquad=\left|\det A_\xi\right|^{1-1/s}a(b)\left[\overline{\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)}+\varepsilon(b,\xi)\right],
\end{aligned}
$$

where

$$
\varepsilon(b,\xi)=\varepsilon_1(b,\xi)+\varepsilon_2(b,\xi)+\varepsilon_3(b,\xi),
$$

and

$$
\begin{aligned}
\varepsilon_1(b,\xi)&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{a(b)}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\,(\nabla a)(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx,\\
\varepsilon_2(b,\xi)&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{a(b)}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{i(\vartheta(x+b)-\vartheta(b))}\, r_1(x)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx,\\
\varepsilon_3(b,\xi)&=(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x)\, r_2(x)\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx\\
&\quad+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{a(b)}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-(\nabla\vartheta)(b))}\gamma(x)\\
&\quad\times r_2(x)(\nabla a)(b)\cdot x\,\overline{\varphi\left(A_\xi^{-1}x\right)}dx.
\end{aligned}
$$

Suppose that the term $\varepsilon(b,\xi)$ is negligible and assume $f(x)=a(x)e^{i\vartheta(x)}$ as before. Then

$$
\begin{aligned}
\left(\mathrm{S}_{s,A,\varphi}f\right)(b,\xi)&\doteqdot e^{-i(b\cdot\xi-\vartheta(b))}\left|\det A_\xi\right|^{1-1/s}a(b)\,\overline{\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)}\\
&=\left|\det A_\xi\right|^{1-1/s}e^{-ib\cdot\xi}f(b)\,\overline{\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)},
\end{aligned}
$$

and the multi-dimensional energy distibution for the Stockwell transform is given by

$$
\left|\left(\mathrm{S}_{s,A,\varphi}f\right)(b,\xi)\right|^2=\left|\det A_\xi\right|^{2-2/s}\left|a(b)\right|^2\left|\widehat{\varphi}\left(A_\xi^t\left((\nabla\vartheta)(b)-\xi\right)\right)\right|^2.
$$

Assume that

$$\sup_{x\in\mathbb{R}^n}\left|\widehat{\varphi}\left(x\right)\right| = \left|\widehat{\varphi}\left(0\right)\right|,$$

then for each fixed b a local-maxima occours at $\xi\left(b\right) = \left(\nabla\vartheta\right)\left(b\right)$. Notice that these points of local maxima do not depend on A_ξ. We call these points $\left(b,\xi\left(b\right)\right)$ ridges and we observe that the ridge frequency $\xi\left(b\right)$ is the instantaneous frequency $\left(\nabla\vartheta\right)\left(b\right)$ at the point b. Let us assume in addition that

$$\widehat{\varphi}\left(x\right) > 0, \qquad \forall x\in\mathbb{R}^n,$$

and write the Stockwell transform in polar coordinates as

$$\left(\mathrm{S}_{s,A,\varphi}f\right)\left(b,\xi\right) = c\left(b,\xi\right)e^{i\Theta\left(b,\xi\right)},$$

then

$$\Theta\left(b,\xi\right) = b\cdot\xi - \vartheta\left(b\right),$$

and the ridges are the points for which $\xi = \nabla\vartheta\left(b\right)$ or, equivalently,

$$\left(\nabla_b\Theta\right)\left(b,\xi\right) = \xi - \left(\nabla\vartheta\right)\left(b\right) = 0.$$

Finally, we can write

$$\left(\nabla\vartheta\right)\left(b\right) = \left(\nabla_b\Theta\right)\left(b,\xi\right) + \xi.$$

Now we can give the hypotheses that guarantee the negligibility of ε.

Lemma 7.2. *Let $f\in L^2\left(\mathbb{R}^n\right)$ be a complex-valued signal. Then we can find $\gamma:\mathbb{R}^n\to\mathbb{C}$, $r_1:\mathbb{R}^n\to\mathbb{C}$ and $r_2:\mathbb{R}^n\to\mathbb{C}$ with r_1, $r_2\in\mathscr{O}\left(|x|^2\right)$, i.e., there exist M_1, M_2, δ_1, $\delta_2>0$ such that*

$$\left|r_j\left(x\right)\right| \leqslant M_j\left|x\right|^2, \qquad \forall x\in B_{\delta_j}\left(0\right), \quad j=1,2,$$

where

$$B_{\delta_j}\left(0\right) = \left\{x\in\mathbb{R}^n : |x| < \delta_j\right\}.$$

Then

$$\left(\mathrm{S}_{s,A,\varphi}f\right)\left(b,\xi\right) = \left|\det A_\xi\right|^{1-1/s}e^{-ib\cdot\xi}f\left(b\right)\left[\overline{\widehat{\varphi}\left(A_\xi^t\left(\left(IFf\right)\left(b\right)-\xi\right)\right)} + \varepsilon\left(b,\xi\right)\right],$$

where

$$\varepsilon(b,\xi) = \varepsilon_1(b,\xi) + \varepsilon_2(b,\xi) + \varepsilon_3(b,\xi),$$

and we have

$$\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{\left|A_\xi^t(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|y\right| \left|\varphi(y)\right| dy \\
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{M_1}{\left|a(b)\right|} \int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_1\right\}} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n \setminus \left\{y:\left|A_\xi y\right|\leqslant\delta_1\right\}} \left|r_1\left(A_\xi y\right)\right| \left|\varphi(y)\right| dy
\end{aligned}$$

and

$$\begin{aligned}
\left|\varepsilon_3(b,\xi)\right| &\leqslant (2\pi)^{-n/2} M_2 \int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} \int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}} r_2\left(A_\xi y\right) \left|\varphi(y)\right| dx \\
&\quad + (2\pi)^{-n/2} M_2 \frac{\left|A_\xi^t(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}} \left|A_\xi y\right|^2 \left|y\right| \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} \frac{\left|A_\xi^t(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}^n \setminus \left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}} \left|r_2\left(A_\xi y\right)\right| \\
&\quad \times \left|y\right| \left|\varphi(y)\right| dy.
\end{aligned}$$

Remark 7.3. Notice that when $n = 1$, $s = 1$ and $A_\xi = 1/\xi$, we get for all $b, \xi \in \mathbb{R}^n$,

$$\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-1/2} \frac{1}{|\xi|} \frac{\left|a'(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}} \left|y\right| \left|\varphi(y)\right| dy \\
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-1/2} \frac{1}{|\xi|^2} \frac{M_1}{\left|a(b)\right|} \int_{\left\{y:\left|y/\xi\right|\leqslant\delta_1\right\}} \left|y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-1/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R} \setminus \left\{y:\left|y/\xi\right|\leqslant\delta_1\right\}} \left|r_1(y/\xi)\right| \left|\varphi(y)\right| dy
\end{aligned}$$

and

$$\left|\varepsilon_3(b,\xi)\right| \leqslant (2\pi)^{-1/2} \frac{1}{|\xi|^2} M_2 \int_{\left\{y:\left|y/\xi\right|\leqslant\delta_2\right\}} \left|y\right|^2 \left|\varphi(y)\right| dy$$

$$+ (2\pi)^{-1/2} \int_{\{y:|y/\xi|\leqslant\delta_2\}} r_2(y/\xi) \left|\varphi(y)\right| dy$$
$$+ (2\pi)^{-1/2} M_2 \frac{1}{|\xi|^3} \frac{\left|a'(b)\right|}{\left|a(b)\right|} \int_{\{y:|y/\xi|\leqslant\delta_2\}} \left|y\right|^3 \left|\varphi(y)\right| dy$$
$$+ (2\pi)^{-1/2} \frac{1}{|\xi|} \frac{\left|a'(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}\setminus\{y:|y/\xi|\leqslant\delta_2\}} \left|r_2(y/\xi)\right|\left|y\right|\left|\varphi(y)\right| dy.$$

Proof. We can estimate ε_1, ε_2, ε_3 via direct computation and obtain

$$\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|(\nabla a)(b)\cdot x\right| \left|\overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&= (2\pi)^{-n/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|(\nabla a)(b)\cdot A_\xi y\right| \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|A_\xi^t (\nabla a)(b)\cdot y\right| \left|\varphi(y)\right| dy \\
&\leqslant (2\pi)^{-n/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|A_\xi^t (\nabla a)(b)\right| \left|y\right| \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} \frac{\left|A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|y\right| \left|\varphi(y)\right| dy,
\end{aligned}$$

$$\begin{aligned}
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n} \left|r_1(x)\, \overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{B_{\delta_1}(0)} \left|r_1(x)\right| \left|\overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&\quad + (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n\setminus B_{\delta_1}(0)} \left|r_1(x)\, \overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&= (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{B_{\delta_1}(0)} M_1 |x|^2 \left|\overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&\quad + (2\pi)^{-n/2} \left|\det A_\xi\right|^{-1} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n\setminus B_{\delta_1}(0)} \left|r_1(x)\, \overline{\varphi\left(A_\xi^{-1}x\right)}\right| dx \\
&= (2\pi)^{-n/2} \frac{M_1}{\left|a(b)\right|} \int_{\{y:|A_\xi y|\leqslant\delta_1\}} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} \frac{1}{\left|a(b)\right|} \int_{\mathbb{R}^n\setminus\{y:|A_\xi y|\leqslant\delta_1\}} \left|r_1\left(A_\xi y\right)\right| \left|\varphi(y)\right| dy
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\varepsilon_3(b,\xi)\right| \\
&\leqslant (2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{\mathbb{R}^n}\left|r_2(x)\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{\left|a(b)\right|}\int_{\mathbb{R}^n}\left|r_2(x)\right|\left|(\nabla a)(b)\cdot x\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&\leqslant (2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{B_{\delta_2}(0)}\left|r_2(x)\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{\mathbb{R}^n\setminus B_{\delta_2}(0)}\left|r_2(x)\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{\left|a(b)\right|}\int_{B_{\delta_2}(0)}\left|r_2(x)\right|\left|(\nabla a)(b)\cdot x\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{\left|a(b)\right|}\int_{\mathbb{R}^n\setminus B_{\delta_2}(0)}\left|r_2(x)\right| \\
&\quad\times\left|(\nabla a)(b)\cdot x\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&\leqslant (2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{B_{\delta_2}(0)}M_2|x|^2\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\int_{\mathbb{R}^n\setminus B_{\delta_2}(0)}\left|r_2(x)\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{M_2}{\left|a(b)\right|}\int_{B_{\delta_2}(0)}|x|^2\left|(\nabla a)(b)\cdot x\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&+(2\pi)^{-n/2}\left|\det A_\xi\right|^{-1}\frac{1}{\left|a(b)\right|}\int_{\mathbb{R}^n\setminus B_{\delta_2}(0)}\left|r_2(x)\right| \\
&\quad\times\left|(\nabla a)(b)\cdot x\right|\left|\varphi\left(A_\xi^{-1}x\right)\right|dx \\
&\leqslant (2\pi)^{-n/2}M_2\int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}}\left|A_\xi y\right|^2\left|\varphi(y)\right|dy \\
&+(2\pi)^{-n/2}\int_{\mathbb{R}^n\setminus\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}}r_2\left(A_\xi y\right)\left|\varphi(y)\right|dy \\
&+(2\pi)^{-n/2}\frac{M_2}{\left|a(b)\right|}\int_{\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}}\left|A_\xi y\right|^2 \\
&\quad\times\left|(\nabla a)(b)\cdot\left(A_\xi y\right)\right|\left|\varphi(y)\right|dy \\
&+(2\pi)^{-n/2}\frac{1}{\left|a(b)\right|}\int_{\mathbb{R}^n\setminus\left\{y:\left|A_\xi y\right|\leqslant\delta_2\right\}}\left|r_2\left(A_\xi y\right)\right| \\
&\quad\times\left|(\nabla a)(b)\cdot\left(A_\xi y\right)\right|\left|\varphi(y)\right|dy
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant (2\pi)^{-n/2} M_2 \int_{\{y:|A_\xi y|\leqslant\delta_2\}} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&+ (2\pi)^{-n/2} \int_{\mathbb{R}^n\setminus\{y:|A_\xi y|\leqslant\delta_2\}} r_2\left(A_\xi y\right) \left|\varphi(y)\right| dx \\
&+ (2\pi)^{-n/2} M_2 \frac{\left|A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\{y:|A_\xi y|\leqslant\delta_2\}} \left|A_\xi y\right|^2 \left|y\right| \left|\varphi(y)\right| dy \\
&+ (2\pi)^{-n/2} \frac{\left|A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\mathbb{R}^n\setminus\{y:|A_\xi y|\leqslant\delta_2\}} \left|r_2\left(A_\xi y\right)\right| \left|y\right| \left|\varphi(y)\right| dy.
\end{aligned}
$$

Proposition 7.4. *Let $f \in L^2(\mathbb{R}^n)$ be a complex-valued signal. Then we can find $\gamma : \mathbb{R}^n \to \mathbb{C}$, $r_1 : \mathbb{R}^n \to \mathbb{C}$ and $r_2 : \mathbb{R}^n \to \mathbb{C}$ with r_1, $r_2 \in \mathcal{O}\left(|x|^2\right)$, i.e., we can find M_1, M_2, δ_1, $\delta_2 > 0$ such that*

$$
\left|r_j(x)\right| \leqslant M_j |x|^2, \qquad \forall x \in B_{\delta_j}(0), \quad j = 1, 2,
$$

where

$$
B_{\delta_j}(0) = \left\{x \in \mathbb{R}^n : |x| < \delta_j\right\}.
$$

Then

$$
\left(\mathrm{S}_{s,A,\varphi} f\right)(b,\xi) = \left|\det A_\xi\right|^{1-1/s} e^{-ib\cdot\xi} f(b) \left[\overline{\widehat{\varphi}\left(A_\xi^t\left((IFf)(b) - \xi\right)\right)} + \varepsilon(b,\xi)\right],
$$

where

$$
\varepsilon(b,\xi) = \varepsilon_1(b,\xi) + \varepsilon_2(b,\xi) + \varepsilon_3(b,\xi).
$$

Let δ be the minimum between δ_1 and δ_2, let M be the maximum between M_1 and M_2. Furthermore assume that there exists

$$
0 < \rho < \frac{\delta}{\left\|A_\xi\right\|}, \qquad \forall \xi \in \mathbb{R}^n,
$$

where

$$
\left\|A_\xi\right\| = \sup_{y\neq 0} \frac{\left|A_\xi y\right|}{\left|y\right|} < \infty.
$$

Let φ be a compactly supported function such that

$$\operatorname{supp}\varphi \subset B_\rho(0).$$

Then

$$\left|\varepsilon_1(b,\xi)\right| \leqslant (2\pi)^{-n/2} \frac{\left|A_\xi^t(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{B_\rho(0)} |y|\left|\varphi(y)\right| dy,$$
$$\left|\varepsilon_2(b,\xi)\right| \leqslant (2\pi)^{-n/2} \frac{M}{\left|a(b)\right|} \int_{\{y:|A_\xi y|\leqslant\delta_1\}\cap B_\rho(0)} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy.$$

and

$$\begin{aligned}
&\left|\varepsilon_3(b,\xi)\right| \\
&\leqslant (2\pi)^{-n/2} M \int_{\{y:|A_\xi y|\leqslant\delta_1\}\cap B_\rho(0)} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&+ (2\pi)^{-n/2} M \frac{\left|A_\xi^t(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{\{y:|A_\xi y|\leqslant\delta_1\}\cap B_\rho(0)} \left|A_\xi y\right|^2 |y|\left|\varphi(y)\right| dy.
\end{aligned}$$

Proof. The estimate is trivial for ε_1. Via direct computation we can work on ε_2. Notice that since $A_\xi \in \mathrm{GL}(n,\mathbb{R})$

$$\left\|A_\xi\right\| = \sup_{y\neq 0}\left|A_\xi y\right| < \infty,$$

so

$$\begin{aligned}
\mathbb{R}^n \setminus \{y : |A_\xi y| \leqslant \delta_1\} &= \{y : |A_\xi y| > \delta_1\} \subset \{y : \|A_\xi\|\,|y| > \delta_1\} \\
&= \left\{y : |y| > \frac{\delta_1}{\|A_\xi\|}\right\} \subset \left\{y : |y| > \frac{\delta}{\|A_\xi\|}\right\}.
\end{aligned}$$

Since

$$\rho < \frac{\delta}{\|A_\xi\|}, \qquad \forall \xi \in \mathbb{R}^n,$$

we have

$$\begin{aligned}
&= \{y : |y| \leqslant \rho\} \cap \mathbb{R}^n \setminus \{y : |A_\xi y| \leqslant \delta_1\} \{y : |y| \leqslant \rho\} \cap \{y : |A_\xi y| > \delta_1\} \\
&\subset \left\{y : |y| \leqslant \frac{\delta}{\|A_\xi\|}\right\} \cap \left\{y : |y| > \frac{\delta}{\|A_\xi\|}\right\} = \varnothing.
\end{aligned}$$

So

$$
\begin{aligned}
&\left|\varepsilon_2(b,\xi)\right| \\
&\leqslant (2\pi)^{-n/2}\frac{M_1}{\left|a(b)\right|}\int_{\{y:|A_\xi y|\leqslant\delta_1\}}\left|A_\xi y\right|^2\left|\varphi(y)\right|dy \\
&\quad + (2\pi)^{-n/2}\frac{1}{\left|a(b)\right|}\int_{\mathbb{R}^n\setminus\{y:|A_\xi y|\leqslant\delta_1\}}\left|r_1(A_\xi y)\right|\left|\varphi(y)\right|dy \\
&= (2\pi)^{-n/2}\frac{M_1}{\left|a(b)\right|}\int_{\{y:|A_\xi y|\leqslant\delta_1\}\cap\{y:|y|<\rho\}}\left|A_\xi y\right|^2\left|\varphi(y)\right|dy \\
&\quad + (2\pi)^{-n/2}\frac{1}{\left|a(b)\right|}\int_{(\mathbb{R}^n\setminus\{y:|A_\xi y|\leqslant\delta_1\})\cap\{y:|y|<\rho\}}\left|r_1(A_\xi y)\right|\left|\varphi(y)\right|dy \\
&= (2\pi)^{-n/2}\frac{M_1}{\left|a(b)\right|}\int_{\{y:|A_\xi y|\leqslant\delta_1\}\cap\{y:|y|<\rho\}}\left|A_\xi y\right|^2\left|\varphi(y)\right|dy.
\end{aligned}
$$

Analoguosly we can deal with ε_3.

7.2 Examples

7.2.1 1-dimensional case

Let $n=1$, and $A:\mathbb{R}\ni\xi\mapsto A_\xi\in\mathbb{R}\setminus(0)$. Then for all $\left|A_\xi\right|^{-1}>\rho/\delta_1$

$$
\{y:\left|A_\xi y\right|\leqslant\delta_1\}\cap B_\rho(0)=B_\rho(0),
$$

and, in the same setting as Proposition 7.4, we have

$$
\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-1/2}\left|A_\xi\right|\frac{\left|(a')(b)\right|}{\left|a(b)\right|}\int_{B_\rho(0)}\left|y\right|\left|\varphi(y)\right|dy, \\
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-1/2}\left|A_\xi\right|^2\frac{M}{\left|a(b)\right|}\int_{B_\rho(0)}\left|y\right|^2\left|\varphi(y)\right|dy.
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\varepsilon_3(b,\xi)\right| \leqslant{}& (2\pi)^{-1/2}\left|A_\xi\right|^2 M\int_{B_\rho(0)}\left|y\right|^2\left|\varphi(y)\right|dy \\
&+ (2\pi)^{-1/2}\left|A_\xi\right|^3 M\frac{\left|(\nabla a)(b)\right|}{\left|a(b)\right|}\int_{B_\rho(0)}\left|y\right|^3\left|\varphi(y)\right|dy.
\end{aligned}
$$

Choosing $s=1$ and $A_\xi=1/|\xi|$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$, we can obtain similar results as the ones in [19].

7.2.2 Diagonal matrices

In this subsection we study the estimates given in Proposition 7.4 for diagonal matrices as in Section 5.2. In this case we have

$$\left|\det A_\xi\right| = \prod_{i=1}^{n} \frac{1}{|\xi_i|}.$$

Then

$$\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{\left|A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{B_\rho(0)} \left|y\right|\left|\varphi(y)\right| dy,\\
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{M}{\left|a(b)\right|} \int_{\left\{y:\left|A_\xi y\right|\leqslant \delta_1\right\}\cap B_\rho(0)} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy.
\end{aligned}$$

and

$$\begin{aligned}
&\left|\varepsilon_3(b,\xi)\right|\\
&\leqslant (2\pi)^{-n/2} M \int_{\left\{y:\left|A_\xi y\right|\leqslant \delta_1\right\}\cap B_\rho(0)} \left|A_\xi y\right|^2 \left|\varphi(y)\right| dy\\
&+ (2\pi)^{-n/2} M \frac{\left|A_\xi^t (\nabla a)(y)\right|}{\left|a(y)\right|} \int_{\left\{y:\left|A_\xi y\right|\leqslant \delta_1\right\}\cap B_\rho(0)} \left|A_\xi y\right|^2 \left|y\right|\left|\varphi(y)\right| dy.
\end{aligned}$$

7.2.3 Rotation matrices in dimension $n = 1, 2, 4, 8$

In this subsection we study the estimates given in Proposition 7.4 for diagonal matrices as in Section 5.3. In this case we have

$$\left|\det A_\xi\right| = |\xi|^{-n}.$$

Then

$$\begin{aligned}
\left|\varepsilon_1(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{\left|\frac{|\xi|}{|\xi|} A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{B_\rho(0)} \left|y\right|\left|\varphi(y)\right| dy\\
&= (2\pi)^{-n/2} \frac{\left||\xi| A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \int_{B_\rho(0)} \left|y\right|\left|\varphi(y)\right| dy
\end{aligned}$$

$$= (2\pi)^{-n/2} |\xi|^{-1} \frac{\left|(\nabla a)(b)\right|}{\left|a(b)\right|} \int_{B_\rho(0)} \left|y\right| \left|\varphi(y)\right| dy$$

$$\begin{aligned}
\left|\varepsilon_2(b,\xi)\right| &\leqslant (2\pi)^{-n/2} \frac{M}{\left|a(b)\right|} \int_{\left\{y:\left|\frac{|\xi|}{|\xi|}A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} \left|\frac{|\xi|}{|\xi|}A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} \frac{M}{\left|a(b)\right|} \int_{\left\{y:|\xi|^{-1}\left||\xi|A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} |\xi|^{-2} \left||\xi| A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} |\xi|^{-2} \frac{M}{\left|a(b)\right|} \int_{\left\{y:\left|y\right|\leqslant|\xi|\delta_1\right\}\cap B_\rho(0)} \left|y\right|^2 \left|\varphi(y)\right| dy,
\end{aligned}$$

and

$$\begin{aligned}
\left|\varepsilon_3(b,\xi)\right| &\leqslant (2\pi)^{-n/2} M \int_{\left\{y:\left|\frac{|\xi|}{|\xi|}A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} \left|\frac{|\xi|}{|\xi|}A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} M \frac{\left|\frac{|\xi|}{|\xi|}A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \\
&\quad \times \int_{\left\{y:\left|\frac{|\xi|}{|\xi|}A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} \left|\frac{|\xi|}{|\xi|}A_\xi y\right|^2 \left|y\right| \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} M \int_{\left\{y:|\xi|^{-1}\left||\xi|A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} |\xi|^{-2} \left||\xi| A_\xi y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} M \frac{|\xi|^{-1}\left||\xi| A_\xi^t (\nabla a)(b)\right|}{\left|a(b)\right|} \\
&\quad \times \int_{\left\{y:|\xi|^{-1}\left||\xi|A_\xi y\right|\leqslant\delta_1\right\}\cap B_\rho(0)} |\xi|^{-2} \left||\xi| A_\xi y\right|^2 \left|y\right| \left|\varphi(y)\right| dy \\
&= (2\pi)^{-n/2} |\xi|^{-2} M \int_{\left\{y:\left|y\right|\leqslant|\xi|\delta_1\right\}\cap B_\rho(0)} \left|y\right|^2 \left|\varphi(y)\right| dy \\
&\quad + (2\pi)^{-n/2} |\xi|^{-3} M \frac{\left|(\nabla a)(y)\right|}{\left|a(y)\right|} \int_{\left\{y:\left|y\right|\leqslant|\xi|\delta_1\right\}\cap B_\rho(0)} \left|y\right|^3 \left|\varphi(y)\right| dy
\end{aligned}$$

Notice that, for all $|\xi| > \frac{\rho}{\delta_1}$, we get

$$\left|\varepsilon_2(b,\xi)\right| \leqslant (2\pi)^{-n/2} |\xi|^{-2} \frac{M}{\left|a(b)\right|} \int_{B_\rho(0)} \left|y\right|^2 \left|\varphi(y)\right| dy,$$

and

$$\left|\varepsilon_3(b,\xi)\right| \leqslant (2\pi)^{-n/2} |\xi|^{-2} M \int_{B_\rho(0)} \left|y\right|^2 \left|\varphi(y)\right| dy$$

$$+ (2\pi)^{-n/2} |\xi|^{-3} M \frac{\left|(\nabla a)(y)\right|}{\left|a(y)\right|} \int_{B_\rho(0)} \left|y\right|^3 \left|\varphi(y)\right| dy.$$

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